BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 3, May 1976

ON THE NUMBER OF INVARIANT CLOSED GEODESICS

BY KARSTEN GROVE AND MINORU TANAKA

Communicated by Shing S. Chern, January 19, 1976

It is an outstanding problem in riemannian geometry whether any compact riemannian manifold of dimension $n + 1 > 1$ has infinitely many closed geodesics. In this note we outline a proof of the following:

THEOREM. *Let M be a compact, 1-connected riemannian manifold and $A: M \longrightarrow M$ an isometry of finite order. Then A has infinitely many closed invariant geodesics if the sequence of Betti numbers for the space of maps $\sigma: \mathbf{R} \longrightarrow M$ with $\sigma(t + 1) = A(\sigma(t))$ is unbounded.*

This is a generalization of a well-known theorem on closed geodesics ($A = 1_M$) by Gromoll and Meyer [2]. Observe that the assumption on the Betti numbers in our theorem is essential (A = rotation on S^2). Note also that the isometries of finite order are dense in the isometry group.

OUTLINE OF PROOF. Let $\Lambda(M, A)$ be the complete, riemannian Hilbert manifold of all absolutely continuous maps $\sigma: \mathbf{R} \longrightarrow M$ with $\dot{\sigma}: \mathbf{R} \longrightarrow TM$ locally square integrable and $\sigma(t + 1) = A(\sigma(t))$ [4]. The critical points for the energy integral $E^A: \Lambda(M, A) \longrightarrow \mathbf{R}$ correspond to A-invariant geodesics, and E^A satisfies condition (C) of Palais and Smale [4]. The fixed point set of A, $\mathrm{Fix}(A)$ corresponds to the critical points with E^A-value zero, and it consists of finitely many nondegenerate critical submanifolds of $\Lambda(M, A)$. The contribution of $\mathrm{Fix}(A)$ to the homology of $\Lambda(M, A)$ is therefore at most finite dimensional.

The $\mathbf{R}$-action on $\Lambda(M, A)$ induced by translation of the parameter reduces to an $S^1 = \mathbf{R}/s \cdot \mathbf{Z}$-action, when A has order $s \in \mathbf{Z}^+$. If γ is a nontrivial closed A-invariant geodesic, it is represented by a critical point $c \in \Lambda(M, A)$ whose fundamental period is s/m for some integer $m \leqslant s$. Let $s/m = s_0/m_0$, where s_0 and m_0 are relatively prime positive integers, and choose integers n_0 and k_0 such that $m_0 n_0 = 1 + s_0 k_0$. Define $c^u: \mathbf{R} \longrightarrow M$ for any $u \in \mathbf{R}$ by $c^u(t) = c(u \cdot t)$ and put $\bar{c} = c^{1/m_0}$. Then $\bar{c}$ is a critical point for $E^{A^{n_0}}$ with fundamental period s_0 and $\bar{c} \subset \mathrm{Fix}(A^{s_0})$. For any integers m and r with $ms_0 + rm_0 \neq 0$, $\bar{c}^{ms_0 + rm_0}$ is a critical point for E^{A^r} and $S^1 \cdot c^{ms_0 + m_0}$, $m \in \mathbf{Z}^+ \cup \{0\}$ are all the critical orbits in $\Lambda(M, A)$ "generated" by γ. In analogy to Bott [1] we find formulas for the indices and nullities of the critical orbits $S^1 \cdot \bar{c}^{ms_0 + rm_0}$ in $\Lambda(M, A^r)$ from which we derive:

AMS (MOS) subject classifications (1970). Primary 58E10; Secondary 53C20.

Key words and phrases. Isometry-invariant geodesic, equivariant degenerate Morse theory, index, nullity, characteristic invariant.