BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, March 1976.

ON THE SELBERG TRACE FORMULA IN THE CASE OF COMPACT QUOTIENT

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1. **Introduction.** Let G be a connected unimodular Lie group. Let Γ be a discrete subgroup of G so that $\Gamma\backslash G$ is compact. We fix a Haar measure, dg, on G. Then dg induces a G-invariant measure on $\Gamma\backslash G$. We can then form a unitary representation $(\pi_\Gamma, L^2(\Gamma\backslash G))$ where $(\pi_\Gamma(g)f)(x)=f(xg)$ for $f\in L^2(\Gamma\backslash G)$, $x\in\Gamma\backslash G$, $g\in G$. If $\phi\in C_c^\infty(G)$ (the space of all C^∞ compactly supported complex valued functions on G) we can form

$$(\pi_\Gamma(\phi)f)(x)=\int_G \phi(g)f(xg)\,dg.$$

It is a standard fact (see §2) that $\pi_\Gamma(\phi)$ is of trace class. In particular, $\pi_\Gamma(\phi)$ is completely continuous for $\phi\in C_c^\infty(G)$. This implies that $L^2(\Gamma\backslash G)$ decomposes into an orthogonal direct sum of irreducible invariant subspaces, $\{H_j\}_{j=1}^\infty$ and for each i there are only a finite number of k so that H_j is equivalent with H_k as a representation of G (cf. Gelfand, Graev, Pyateckiĭ-Shapiro [**9**]). Let $\hat G$ denote the set of equivalence classes of irreducible representations of G. Then we have observed that

$$\pi_\Gamma=\sum_{\omega\in\hat G} N_\Gamma(\omega)\omega$$

where $N_\Gamma(\omega)$ is a nonnegative integer. If $\omega\in\hat G$ we say that ω is of trace class if for each $(\pi, H)\in\omega$, $\phi\in C_c^\infty(G)$, $\pi(\phi)=\int_G\phi(g)\pi(g)\,dg$ is a trace class operator on H. If $\omega\in\hat G$ is of trace class, then set $\Theta_\omega(\phi)=\operatorname{tr}\pi(\phi)$ for $(\pi, H)\in\omega$. The above observations imply that if $\omega\in\hat G$ and $N_\Gamma(\omega)\neq 0$, then ω is of trace class. We therefore see that if $\phi\in C_c^\infty(G)$, then

$$\operatorname{tr}\pi_\Gamma(\phi)=\sum_{\omega\in\hat G} N_\Gamma(\omega)\Theta_\omega(\phi).$$

The numbers $N_\Gamma(\omega)$ have been the subject of a great deal of investigation in the last few years. In this article we will give a short survey of various techniques that have been used to study these integers. We will concentrate our attention on semisimple Lie groups, G. We will also, for most of the article, look at the easiest groups Γ. These groups have no elements of finite order other than the identity. Without this assumption many (interesting)

AMS(MOS) subject classifications (1970). Primary 10D20, 22E40, 43A15.
This address was given at the annual meeting of the American Mathematical Society in Washington, D.C. on January 24, 1975; received by the editors May 8, 1975.