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THE TOTAL CURVATURE OF KNOTTED SPHERES

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Chern and Lashof [1] proved several inequalities concerning the total curvature of an immersed manifold. Their second result is a weak generalization of the Fary-Milnor theorem [2], [5] for closed space curves. In this paper, a stronger result (Corollary 1), the complete homotopy extension, is stated and proved. I would like to thank Bill Pohl for conversations surrounding the formulation and proof of this result.

I. **Background.** Let $x: M^n \longrightarrow E^{n+N}$ be a C^∞-immersion into Euclidean space of dimension $n + N$ ($N > 0$); and B_ν be the bundle of unit normal vectors of $x(M^n)$. A point of B_ν is a pair $(p, \nu(p))$, where $\nu(p)$ is a unit normal vector to $x(M^n)$ at $x(p)$. The map $\bar{\nu}: B_\nu \longrightarrow S_0^{n+N-1}$, into the unit sphere of E^{n+N}, is defined by $\bar{\nu}(p, \nu(p)) = \nu(p)$.

The Lipschitz-Killing curvature [1], $G(p, \nu)$ at $\nu(p)$, is then given by the $\bar{\nu}$-ratio of corresponding volume elements in S_0^{n+N-1} and B_ν. The *total curvature of M^n at p* is $K^*(p) = \int |G(p, \nu)|\, d\sigma$, the integral being taken over the sphere of unit normal vectors at $x(p)$. The *total curvature of M^n* is given by $K^* = K^*(M) = \int_{p \in M} K^*(p)\, dV$.

The first two Chern-Lashof theorems can be stated as: Given M^n compact without boundary, and $c(m)$ the area of the unit hypersphere $S_0^m \subset E^{m+1}$, then:

COROLLARY 1. $K^*(M) \geqslant 2c(n + N - 1)$.

COROLLARY 2. *If $K^*(M) < 3c(n + N - 1)$, then M is homeomorphic to S^n.*

The essential argument of their proof can be summarized as a lemma.

LEMMA 1. *If, for almost all $v_0 \in S_0^{n+N-1}$, the height function $\langle v_0, - \rangle$: $x(M) \longrightarrow R$ has at least k distinct critical points, then $K^*(M) \geqslant kc(n + N - 1)$.*

Their method is an adaptation of the technique used by Fenchel [3]. This fact suggested that Corollary 2 is a weak generalization of Fary-Milnor.

II. **The main result.** In this section, a curvature inequality is given which distinguishes between different knottings of S^n. The method, based on Chern-Lashof, takes off from a remark of Fox [4] in which P. L. approximations yield the corresponding S^1 result.

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