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COBORDISM FOR POINCARÉ DUALITY GROUPS

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1. **Relative homology for pairs.** Homology and cohomology for a pair of groups $G \supset S$ (cf. [6]) can be extended to pairs (G, S) consisting of a group G and a family of subgroups $S = \{S_i\}$, as follows: If $S = \emptyset$ one takes the usual (absolute) groups of G. If $S \neq \emptyset$, let Δ be the kernel of the G-homomorphism $\bigoplus_i \mathbf{Z}(G/S_i) \twoheadrightarrow \mathbf{Z}$ given by augmentations; A being a G-module, we put $H^k(G, S; A) = H^{k-1}(G; \operatorname{Hom}(\Delta, A))$ and $H_k(G, S; A) = H_{k-1}(G; \Delta \otimes A)$ where G acts diagonally in $\operatorname{Hom}(\Delta, A)$ and $\Delta \otimes A$. One has exact sequences

$$\cdots \longrightarrow H^k(G, S; A) \longrightarrow H^k(G; A) \longrightarrow \prod_i H^k(S_i; A) \xrightarrow{\delta} H^{k+1}(G, S; A) \longrightarrow \cdots,$$

$$\cdots \longrightarrow H_{k+1}(G, S; A) \xrightarrow{\partial} \bigoplus_i H_k(S_i; A) \longrightarrow H_k(G; A) \longrightarrow H_k(G, S; A) \longrightarrow \cdots.$$

2. **Poincaré duality pairs.** The product structure for Ext^* and Tor_* (cf. [4, Chapter XI]) yields, for $\alpha \in H_n(G, S; B)$, cap-products $\alpha \cap -: H^k(G; A) \longrightarrow H_{n-k}(G, S; B \otimes A)$ and $H^k(G, S; A) \longrightarrow H_{n-k}(G; B \otimes A)$, with diagonal G-action in $B \otimes A$.

(1) DEFINITION. (G, S) is a *Poincaré duality pair of dimension n* (in short: PD^n-pair) if there is a G-module structure $\widetilde{\mathbf{Z}}$ on $\mathbf{Z}$ and an element $e \in H_n(G, S; \widetilde{\mathbf{Z}})$ such that $e \cap -: H^k(G; A) \longrightarrow H_{n-k}(G, S; \widetilde{A})$ is an isomorphism for all A and k.

$\widetilde{\mathbf{Z}}$ is called the "dualizing" module. $\widetilde{A}$ stands for $\widetilde{\mathbf{Z}} \otimes A$. If S is empty, (1) coincides with the usual definition (cf. [1]) of a *Poincaré duality group* of dimension n (in short: PD^n-group).

(2) THEOREM. *(G, S) is a PD^n-pair with dualizing module $\widetilde{\mathbf{Z}}$ if and only if G is a duality group (cf. [3]) with dualizing module $\widetilde{\Delta}$.*

Thus various results and criteria for duality groups can be applied to PD^n-pairs; note that $\widetilde{\Delta}$ is $\mathbf{Z}$-free. We give here, and in §3, a list of consequences of (1): Definition (1) is equivalent with $e \cap -: H^k(G, S; A) \longrightarrow H_{n-k}(G; \widetilde{A})$ being an isomorphism for all A and k. The module $\widetilde{\mathbf{Z}}$ and the dimension n are determined by the pair (G, S). The pair is called *orientable* if G acts trivially on $\widetilde{\mathbf{Z}}$,

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