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HIGHER WHITEHEAD GROUPS AND STABLE HOMOTOPY

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ABSTRACT. Comparison of stable homotopy of B_G, algebraic K-theory of $\mathbf{Z}[G]$ and higher Whitehead groups of G. Computations of $\mathrm{Wh}_2(G)$.

In their work on pseudo-isotopy Hatcher and Wagoner [1] defined an obstruction group $\mathrm{Wh}_2(G)$ as follows. Let $\mathrm{GL}(\mathbf{Z}[G])$ (resp. $E(\mathbf{Z}[G])$, resp. $\mathrm{St}(\mathbf{Z}[G])$) be the general linear group (resp. its commutator subgroup, resp. the Steinberg group) of the group algebra $\mathbf{Z}[G]$. The kernel of the natural homomorphism $\mathrm{St}(\mathbf{Z}[G]) \longrightarrow E(\mathbf{Z}[G])$ is the group $K_2(\mathbf{Z}[G])$ defined by Milnor [4]. As usual $\widetilde{K}_2(\mathbf{Z}[G]) = \mathrm{Coker}(K_2(\mathbf{Z}) \longrightarrow K_2(\mathbf{Z}[G]))$.

If x_{ij}^a ($i \neq j$, $a \in \mathbf{Z}[G]$) are the classical generators of $\mathrm{St}(\mathbf{Z}[G])$ one denotes by $W(\pm G)$ the subgroup of $\mathrm{St}(\mathbf{Z}[G])$ generated by the elements $w_{ij}(\pm g) = x_{ij}^{\pm g} x_{ji}^{\mp g^{-1}} x_{ij}^{\pm g}$ ($g \in G$). With these notations the second-order Whitehead group is

$$\mathrm{Wh}_2(G) = K_2(\mathbf{Z}[G])/K_2(\mathbf{Z}[G]) \cap W(\pm G).$$

Let B_G be the classifying space of the (discrete) group G. The nth stable homotopy group of B_G, $\pi_n^s(B_G) = \varinjlim \pi_{n+k}(S^k B_G)$ is also equal to the nth reduced homology group of B_G with coefficients in the sphere spectrum $\mathbf{S}$: $\widetilde{h}_n(B_G; \mathbf{S})$. Equivalently $\pi_n^s(B_G)$ is equal to $\pi_n(\Omega^\infty S^\infty B_G)$ where $\Omega^\infty S^\infty B_G = \varinjlim \Omega^k S^k B_G$.

THEOREM. *There exists a natural homomorphism* $\pi_2^s(B_G) \longrightarrow \widetilde{K}_2(\mathbf{Z}[G])$ *such that* $\mathrm{Wh}_2(G) = \mathrm{Coker}(\pi_2^s(B_G) \longrightarrow \widetilde{K}_2(\mathbf{Z}[G]))$.

Let $\mathbf{K}_A$ be the spectrum of algebraic K-theory associated to the (unitary) ring A (cf. [2], [6]). The homotopy groups $\pi_n(\mathbf{K}_A)$ are Quillen's K-groups denoted $K_n(A) = \pi_n(B^+_{\mathrm{GL}(A)})$, $n \geq 1$ (cf. [5]).

Let $h_*(-; \mathbf{K}_\mathbf{Z})$ be the generalised homology theory associated to $\mathbf{K}_\mathbf{Z}$. We construct natural maps of spectra

$$\mu: \mathbf{S} \longrightarrow \mathbf{K}_\mathbf{Z} \quad \text{and} \quad \lambda: B_G \wedge \mathbf{K}_\mathbf{Z} \longrightarrow \mathbf{K}_{\mathbf{Z}[G]}.$$

These maps give rise to the composed homomorphism

$$\pi_n^s(B_G \cup \mathrm{pt}) \xrightarrow{\mu_n} h_n(B_G; \mathbf{K}_\mathbf{Z}) \xrightarrow{\lambda_n} K_n(\mathbf{Z}[G]).$$

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