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A FORBIDDEN SUBSTRUCTURE CHARACTERIZATION OF GAUSS CODES

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Gauss [2, pp. 272, 282–286] considered the following problem. Given a closed curve in the plane which is *normal*, i.e., it has only finitely many self-intersections and these are transverse double points. Label the crossing points of the curve. The *Gauss code* of the curve is the word obtained by proceeding along the curve and noting each crossing point label as it is traversed. In the resulting word, every label occurs exactly twice. The problem is to characterize those words which are Gauss codes. Such words will be called here *realizable*. For a brief history of the work on the problem see [3, pp. 71–73]. In that reference, Grünbaum says, "Solutions of the characterization problem have been found recently (Treybig [5], Marx [4]); however, they are of the same aesthetically rather unsatisfactory character as Mac Lane's criterion for the planarity of graphs. A characterization of Gauss codes in the spirit of the Kuratowski criterion for planarity of graphs is still missing." This work is an attempt to supply the "missing" criterion. The reader must be the judge of the aesthetic merits. Note that our characterization does meet Edmonds' criterion [1] for a "good characterization".

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In the sequel, the symbols that make up a word are called *letters* and are denoted by upper case Roman letters. Words and sequences of consecutive letters within a word are denoted by lower case Greek letters. For our purposes, two cyclic rearrangements of a word are equivalent.

DEFINITIONS (1). Given a word $\omega = A\alpha A\beta$. We define the *vertex split at* A to be the word $\omega_A = \alpha^{-1}\beta$.

(2) Given a word $\omega = A\alpha A\beta$. We define the *loop removal at* A to be the word obtained by deleting A and both occurrences of the letters in α.

(3) A *subword* of a word ω is any word obtained by a sequence of vertex splits and loop removals.

THEOREM. *A word ω is realizable if and only if it contains no subword of the form*

$$A_1A_2 \cdots A_nA_1A_2 \cdots A_n, \quad n \text{ even}.$$

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