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FLAT HOMOLOGY

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In this note we define "homology groups" relative to the flat site, and list some of their properties, in the case that the base scheme is algebraic over a field.

X_{fl} denotes the big f.p.p.f. site over a scheme X and $S(X_{fl})$ the corresponding category of sheaves. $S = \operatorname{spec} k$, where k is a field of characteristic p. $\mathsf{A}(al)$ denotes the category of commutative algebraic group schemes over S and $\mathsf{A}(u, f) \supset \mathsf{A}(u) \supset \mathsf{A}(uf) \supset \mathsf{A}(f)$ the subcategories consisting of those affine groups which are respectively unipotent or finite, unipotent, unipotent and finite, finite. The letter A always stands for one of these categories and Pro-A for the corresponding pro-category. The notations for derived categories are as in [6].

1. THEOREM (Universal Coefficient Theorem). *For any morphism* π: $X \to S$ *of finite type and any* A, *there exists a complex* $L_\bullet(X/S, \mathsf{A})$ *in* $K^-(\text{Pro-}\mathsf{A})$ *such that*:

(a) $L_s(X/S, \mathsf{A})$ *is a projective object, all* s;

(b) $\operatorname{Hom}_{\text{Pro-}\mathsf{A}}(L_\bullet(X/S, \mathsf{A}), N) \xrightarrow{\approx} \mathbf{R}\pi_* N_X$ *in* $D^+(S(S_{fl}))$ *for all* N *in* A.

Moreover, $L_\bullet(X/S, \mathsf{A})$ *is unique, up to isomorphism, in* $K^-(\text{Pro-}\mathsf{A})$.

PROOF. Choose a conservative family of points for X_{fl}, and let $C^\bullet(F)$ be the corresponding Godement resolution of a sheaf F [1, XVII 4.2]. Choose L_s to pro-represent the functor $N \longmapsto \Gamma(X, C^s(N_X))$: $\mathsf{A} \longrightarrow \mathsf{A}b$.

2. COROLLARY. *Write* $H_s(X/S, \mathsf{A})$ *for* $H_s(L_\bullet(X/S, \mathsf{A}))$. *There is a spectral sequence*

$$\operatorname{Ext}^r_{\text{Pro-}\mathsf{A}}(H_s(X/S, \mathsf{A}), N) \Rightarrow H^{r+s}(X_{fl}, N_X) \quad \textit{for all } N \textit{ in } \mathsf{A}.$$

3. DEFINITION. $L_\bullet(X/S, \mathsf{A})$ is the flat homology complex of X/S relative to A, and $H_s(X/S, \mathsf{A})$ is the sth flat homology group.

4. REMARKS. (a) Theorem 1 is basically as conjectured by Grothendieck [5, p. 316].

(b) $L_\bullet(X/S, \mathsf{A})$ and $H_s(X/S, \mathsf{A})$ are covariant functors in X/S.

(c) *If* ω_0: $\mathsf{A}(al) \longrightarrow \mathsf{A}(f)$ is the functor taking a group scheme to its maximal finite quotient, then $\omega_0(L_\bullet(X/S, \mathsf{A}(al))) = L_\bullet(X/S, \mathsf{A}(f))$. Thus there

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