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EXTENSION THEOREMS FOR NONLINEAR DISJOINTLY ADDITIVE FUNCTIONALS AND OPERATORS ON LEBESGUE SPACES, WITH APPLICATIONS

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In recent years there has been developed a reasonably complete representation theory for nonlinear disjointly additive functionals defined on various Banach spaces of measurable functions [1], [6], [7]. Such functionals, which are involved in many nonlinear integral and differential equations, share some important regularity properties of linear functionals [4, Chapter 5]. Hence the question naturally arises as to whether a Hahn-Banach type extension theory is available for disjointly additive functionals . It turns out that a full analogue of the Hahn-Banach theorem does not hold, but there are important classes of subspaces for which an extension result is available. As a byproduct of this extension result one obtains a representation formula for all continuous disjointly additive functionals defined on such subspaces. It should be noted that a representation result of this type *cannot* be obtained by the methods used in the above cited works, since these subspaces are not complete lattices while the completeness of the lattice structure was essential in the arguments used previously. Among the applications stemming from this extension theory is a characterization of those nonlinear functionals on Sobolev spaces $W^p_k(J)$ $(1 \leqslant p \leqslant \infty,\ k \geqslant 1)$ which have the form:

$$N(u) = \int_J H(t, D^k(u(t)))\, dt, \qquad u \in W^p_k(J),$$

where J is a subinterval of the line and $H\colon J \times R \longrightarrow R$ is a function obeying the Carathéodory conditions [5].

Let $(\Omega, \mathcal{T}, m)$ be a finite nonatomic. measure space. We say that a closed subspace $M \subset L^p(m)$ $(1 \leqslant p \leqslant \infty)$ possesses the *d. a. extension property* if every continuous functional $N\colon M \longrightarrow R$ which is disjointly additive in the sense.

$$N(u, v) = N(u) + N(v), \qquad \forall u, v \in M \text{ s.t. } uv = 0,$$

has an extension to $L^p(m)$ which preserves these properties. For the case $p = \infty$, the continuity here is to be taken relative to the topology generated by bounded sequential convergence in measure, in brief the *(bm) topology*. Call a closed linear subspace $M \subset L^\infty(m)$ a *rich subspace* if: (a) M is w^*-closed, (b) M separates

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