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A STRANGE BOUNDED SMOOTH DOMAIN OF HOLOMORPHY

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Introduction. Let $\Omega \subset \mathbf{C}^n$ be a bounded pseudoconvex domain. Does $\overline{\Omega}$ have a neighborhood basis consisting of pseudoconvex domains? It is well known that the answer to this question is, in general, "no". But it has been an open problem, at least since 1933 when the fundamental paper [1] of H. Behnke and P. Thullen appeared, whether the answer might be in the affirmative under the additional hypothesis of smoothness of the boundary $b\Omega$. The main purpose of this note is to give an example of a bounded pseudoconvex domain $\Omega_1 \subset \mathbf{C}^2$ with smooth boundary that nevertheless does not have a Stein neighborhood basis. Additional hypotheses that guarantee the existence of such a basis are given in [3]. The constructed domain Ω_1 has some more strange properties. In particular, the conjecture of R. O. Wells [5, Conjecture 3.1], does not hold for Ω_1 (Theorem 2) and the boundary $b\Omega_1$ cannot be described by a smooth function with positive semidefinite Leviform on the whole $\mathbf{C}^2$ at each point $p \in b\Omega_1$.

Together with the beautiful example of Kohn and Nirenberg [4], this domain Ω_1 shows that bounded smooth domains in $\mathbf{C}^n$ can have quite different analytic properties than strictly pseudoconvex domains.

The proofs of the theorems announced in this note will be contained in a later paper of the authors.

Definition of Ω_1. We fix a smooth function $\lambda: \mathbf{R} \longrightarrow \mathbf{R}^+ \cup \{0\}$ with the properties $\lambda(x) = 0$ for $x \leq 0$, $\lambda''(x) > 0$ for $x > 0$, and such that λ is "sufficiently" convex. For $r > 1$ we define a family of smooth functions ρ: $(\mathbf{C} - \{0\}) \times \mathbf{C} \longrightarrow \mathbf{R}$ by putting

$$\rho_r(z, w) = |w + e^{i \ln z\bar{z}}|^2 - 1 + \lambda(|z|^{-2} - 1) + \lambda(|z|^2 - r^2)$$

and, finally, we take

$$\Omega_{1,r} = \{(z, w) \in (\mathbf{C} - \{0\}) \times \mathbf{C} \mid \rho_r(z, w) < 0\}.$$

The basic properties of the sets $\Omega_{1,r}$ are:

LEMMA. *For each fixed $r > 1$, $\Omega_{1,r}$ is a bounded pseudoconvex domain in $\mathbf{C}^2$ with smooth boundary. The boundary $b\Omega_{1,r}$ is not strictly pseudoconvex exactly on the annulus*

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