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A DIRECTION OF BIFURCATION FORMULA IN THE THEORY OF THE IMMUNE RESPONSE[1]

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In previous work [1], I derived by biological reasoning and mathematical reduction the following system, attributable to G. I. Bell:

$$du/ds = u[\lambda_1 + k\lambda_1 u - k(\alpha_1 - \lambda_1)v + kn\lambda_1 w], \tag{1a}$$

$$dv/ds = \beta\{v[-\lambda_2 - k(\alpha_2 + \lambda_2)u - k\lambda_2 v - kn\lambda_2 w] + k\gamma uw\}, \tag{1b}$$

$$dw/ds = w[-\lambda_3 + k(\alpha_3 - \lambda_3)u - k\lambda_3 v - kn\lambda_3 w - (k\alpha_3/\theta)uw]. \tag{1c}$$

Equations (1) simulate the immune response of an organism to antigen invasion. The dependent variables u, v, w are, respectively the concentrations of antigens, antibodies, and antibody-producing cells. The meanings of all parameters and constants are found in [1, pp. 93–96].

Equations (1) have two nontrivial rest points. The one nearest the origin, (u_f, v_f, w_f), is stable or unstable according to whether $\beta > \beta_c$ or $\beta < \beta_c$, where $\beta_c > 0$ is a critical value of the parameter β in equation (1b). It is shown [1, Theorem 1] that at $\beta = \beta_c$, a continuous family of periodic solutions bifurcates from (u_f, v_f, w_f). I was able to obtain a direction of bifurcation formula only in the special case where $\lambda_3 = 0$. Namely, periodic solutions bifurcate to the left (right) of β_c, and are stable (unstable) if

$$\beta_c > (\alpha_1 - \lambda_1)\lambda_1/((\alpha_1 - \lambda_1)(\alpha_2 + \lambda_2) + 2\lambda_1\lambda_2), \quad (<). \tag{2}$$

Herein I announce the development of a general formula for direction of bifurcation in equations (1), which approaches condition (2) as $\lambda_3 \longrightarrow 0$. An analytic direction of bifurcation formula will be important in developing the global theory of these bifurcated families of periodic solutions, and in ascribing possible biomedical implications. I describe the new formula.

First we substitute $u = u_f + u^0$, $v = v_f + v^0$, $w = w_f + w^0$ into equations (1), and thus obtain equations centered at (u_f, v_f, w_f). Then we let A_{β_c} be the matrix of the linear part of these centered DE's, with $\beta = \beta_c$. The matrix A_{β_c} has the three linearly independent eigenvectors represented symbolically as

$$(\xi_1, \eta_1, \zeta_1), \qquad (\bar{\xi}_1, \bar{\eta}_1, \bar{\zeta}_1), \qquad (\xi, \eta, \zeta). \tag{3}$$

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