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UNIQUE FACTORIZATION IN RANDOM VARIABLES

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The problem of determining the potential $q(x)$ from "spectral data" for the equation

$$-y''(x) + q(x)y(x) = \lambda y(x), \qquad -\infty < x < \infty,$$

has been studied extensively. For a review, see [1] and [5].

A typical kind of result, associated with the names of Gelfand and Levitan tells you that if the discrete spectrum, the normalizing constants, and the reflection coefficient $R(k)$ are known, then $q(x)$ can, in principle, be determined uniquely.

Here we consider a random version of this problem. We envisage a situation where one keeps records of "spectral data" for noisy versions of the potential $q(x)$ and attempts to determine the "mean potential" from the distribution of the data.

It turns out that in this random case a smaller set of quantitites–measured over and over again–give a lot of information about $q(x)$. A similar situation develops in a variety of different setups (see, for instance, [3] and [4]).

Let $-\nabla^2$ stand for the $n \times n$ matrix

$$-\nabla^2 \equiv \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & 0 \\ & -1 & 2 & & \\ & 0 & & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

THEOREM I. *Let $q_1, \ldots, q_n$ be independent Gaussian random variables with unknown means $\bar{q}_1, \ldots, \bar{q}_n$ and variances all equal to one. Then the joint distribution function of* $\operatorname{tr}(-\nabla^2 + qI)^k$, *$k = 1, \ldots, n$, determines the vector $\bar{q}_1, \ldots, \bar{q}_n$ up to a global reflection $\bar{q}'_i \equiv \bar{q}_{n-i+1}$.*

The theorem above says that the spectrum determines the potential up to a trivial reflection. This is to be compared with the nonrandom case where, in

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