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A SMOOTH ACTION ON A SPHERE WITH ONE FIXED POINT

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In this note, we outline the construction of a smooth action of the finite group $SL(2, Z_5)$ on S^7 with exactly one fixed point. Along with some related examples given below, this is the first partial result of a positive nature on the problem of deciding which compact Lie groups can act smoothly on some sphere with one fixed point.

Let S^3 denote the multiplicative group of unit quaternions, and let $G \subset S^3$ be a subgroup isomorphic to $SL(2, Z_5)$, the binary icosahedral group. The space of right cosets, S^3/G, is an integral homology sphere, denoted Σ^3. As pointed out in [2], the natural action of G on Σ^3, although not effective, has exactly one fixed point. We denote this action by λ. Let μ denote the action of G on D^4 given by quaternionic multiplication; this action is free away from the origin. The diagonal action, $\lambda \times \mu$, on $\Sigma^3 \times D^4$ has exactly one fixed point and is free on the boundary.

Now let $f: M^4 \longrightarrow D^4$ denote the collapsing map from the four-dimensional Milnor manifold to a disk. After covering f with a map of stable normal bundles, we obtain a normal map whose surgery obstruction generates the Wall group, $L_0(e)$.

LEMMA. *The surgery obstruction, in $L_3(G)$, of the normal map $f \times$ identity: $M^4 \times \Sigma^3 \longrightarrow D^4 \times \Sigma^3$ is zero.*

Based on this Lemma, the construction is completed as follows. There is a normal cobordism, with the boundary fixed, from $f \times$ identity to $F: W^7 \longrightarrow D^4 \times \Sigma^3$, with F a homotopy equivalence. There is a free action of G on $\widetilde{W}$, the universal cover of W, by covering translations. It is now easy to verify: (1) There is an equivariant diffeomorphism, φ, from the boundary of $\widetilde{W}$ to the restriction of $\lambda \times \mu$ to $\Sigma^3 \times S^3$; (2) The G-manifold obtained by gluing $\widetilde{W}$ to $\Sigma^3 \times D^4$ by φ is a homotopy sphere. Now by using a slight variation of "equivariant connected sum", this action can be modified, away from the single fixed point, so that the ambient manifold is diffeomorphic to S^7.

The proof of the Lemma is quite technical and can only be outlined here. First, the induction theorem of Dress [1] is used to reduce the Lemma to the computation of some surgery obstructions with simpler fundamental groups.

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