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# HYPERBOLIC EQUATIONS AND GROUP REPRESENTATIONS

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I. **Introduction.** In the study of eigenfunction expansions for a differential operator $D$ one usually introduces the resolvent of $D$ or, what is essentially the same thing, the heat operator related to $D$. This means that we study an operator involving more variables than $D$ and then "descend" to $D$ itself. An analogous idea is employed in case $D$ is the Laplacian on the sphere. The eigenfunction theory for $D$ is derived from the study of the Laplacian in the whole euclidean space; "descent" is separation of variables.

Our ideas can be thought of as an extension of separation of variables. Suppose we are given a homogeneous space $V = G/H$ of $G$. We want to decompose the representation of $G$ on $L_2(V)$ (if $V$ has an invariant measure) or on other function spaces on $V$. In rough terms, this is the problem of simultaneous eigenfunction expansion for the operators in the enveloping algebra of $G$ which commute with $H$. The introduction of more variables is accomplished by finding a finite dimensional representation $\rho$ of $G$ which has an orbit which is $G/H$. $\rho$ must be "suitable" in order that we can find a system of differential equations in the whole representation space which descends properly to this orbit. In what follows we illustrate the theory.

II. **Hyperbolicity and symmetric spaces.** Let $G$ be a real semisimple Lie group in Chevalley (normal) form and let $\rho_1, \ldots, \rho_r$ be its fundamental representations. We set $\rho = \rho_1^2 \oplus \ldots \oplus \rho_r^2$. Now for each $i$ there is a point $u_i$ in the representation space of $\rho_i^2$ which is fixed exactly by $K$. All other points which are fixed under $\rho_i^2$ by $K$ are of the form $t_i u_i$ where $t_i$ is a scalar. We call $\{t_1 u_1 + \ldots + t_r u_r\} = T$ the *time axis* in analogy to the case $G = \mathrm{SL}(2, R)$. $T$ is the set of $K$ fixed vectors.

We call $v_i$ the highest weight vector for each $\rho_i^2$ and we set $v = \Sigma v_i$. Then the isotropy group of $v$ is $MN$. We call $\rho(G)v = \Gamma^+$ the *positive light cone.* The real algebraic closure of $\Gamma^+$ is denoted by $\Gamma$ and is called the *light cone.* Note that $A$ normalizes $MN$ so $A$ acts on $\Gamma^+$. This action of $A$ coincides with scalar multiplication, a fact which is crucial in what follows.

Another important property of $\rho$ is that both $\rho(G)u \approx G/K$ and $\Gamma^+ \approx G/MN$ appear in the same representation space. Thus we can study relations be-

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