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ON A MEAN VALUE INEQUALITY

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In this note we discuss a mean value inequality satisfied by functions $u(x, t)$ defined in the half space R_+^{n+1} which are solutions of a partial differential equation of semielliptic type. We then apply this result to the study of spaces of non-isotropic Riesz potentials and to the determination of the classes which arise as traces of the functions $u(x, t)$. The justification for considering these functions lies in the fact that they are a natural substitute for harmonic functions when Laplace's equation is not satisfied and they are related to the study of singular integrals with mixed homogeneity. It is a pleasure to acknowledge the conversations we had with Dr. A. P. Calderón concerning these topics.

The mean value inequality. We let $\{A_t\}_{t>0}$, $A_{ts} = A_tA_s$ be a continuous group of affine transformations of R^n leaving the origin fixed and denote its infinitesimal generator by P so that $t(d/dt)A_t = PA_t$. We further assume that $(Px, x) \geqslant (x, x)$ for $x \in R^n$ and associate to each group A_t a translation invariant distance function $\rho(x)$ defined to be the unique value of t such that $|A_t^{-1}x| = 1$, $\rho(0) = 0$. To the transpose A_t^* of A_t we associate $\rho^*(x)$ in a similar fashion. As is well known $\det A_t = \det A_t^* = t^\gamma$, $\gamma = \operatorname{trace} P$ (see [5, §1.1]). For $\alpha = (\alpha_1, \ldots, \alpha_k)$, $1 \leqslant \alpha_i \leqslant n$, and $x^1, \ldots, x^k$ in R^n we let $\zeta = x^1 \otimes \cdots \otimes x^k$ to be the element with components $\zeta_\alpha = \Pi_{i=1}^k x^i_{\alpha_i}$. For $n \times n$ matrices $A_1, \ldots, A_k$, we put $(A_1 \otimes \cdots \otimes A_k)(x^1 \otimes \cdots \otimes x^k) = A_1x^1 \otimes \cdots \otimes A_kx^k$ and abbreviate this by $\bigotimes^k Ax$ when $A_i = A$, $x^i = x$ for $1 \leqslant i \leqslant k$.

$\partial = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$, $\partial/\partial t$ and $\bigotimes^k A\partial$ acting on functions $u(x, t)$ have the obvious meaning. We set $p_k(t, \partial) = \bigotimes^k LA_t^*\partial$, where $L^2 = (P + P^*)/4\pi$. Given a function $\psi(x)$ we define the dilations $\psi_t(x) = t^{-\gamma}\psi(A_t^{-1}x)$. A special role is played by $\varphi_t(x)$ with $\varphi(x) = e^{-\pi|x|^2}$. This particular function $\varphi_t(x)$ satisfies a differential equation, as is readily seen by taking Fourier transforms, namely $A\varphi_t(x) = 0$ where

$$A = \frac{\partial}{\partial t} - \frac{1}{2\pi t}(P^*A_t^*\partial, A_t^*\partial) = \frac{\partial}{\partial t} - \frac{1}{t}(LA_t^*\partial, LA_t^*\partial).$$

We also have $Au(x, t) = 0$, whenever $u(x, t) = f_*\varphi_t(x)$, $f \in S'(R^n)$.

We now state the mean value inequality and give some applications in the following sections.

MEAN VALUE INEQUALITY. Let $Au(x, t) = 0$ and $0 \leqslant r \leqslant k$, then

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