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# A PRESENTATION FOR SOME $K_2(n, R)$

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1. All rings are commutative with identity. We announce a presentation for the $K_2$ of a class of rings which includes the local ones. We also give a presentation for the relative $K_2$ of a homomorphism that splits and has its kernel in the Jacobson radical. These results generalize (and were suggested by) various earlier ones: the presentation of Matsumoto for the $K_2$ of (infinite) fields [6], [7, §11, 12]; the presentation of Dennis and Stein for the $K_2$ of discrete valuation rings and homomorphic images thereof [2]; stability results of the same authors [4]; the presentation for the relative $K_2$ of dual numbers, by one of us [5]. We reproved most of the earlier results and generalized them in the process.

2. **The functor** $D$ (cf. [3, §9]).

2.1. Let $R$ be a ring, $R^*$ its group of units. We define the abelian group $D(R)$ by the following presentation:

Generators are the symbols $\langle a, b\rangle$ with $a, b \in R$ such that $1 + ab \in R^*$.

Relations are: (D0) commutativity.

(D1) $\langle a, b\rangle \langle -b, -a\rangle = 1$.

(D2) $\langle a, b\rangle \langle a, c\rangle = \langle a, b + c + abc\rangle$.

(D3) $\langle a, bc\rangle = \langle ab, c\rangle \langle ac, b\rangle$.

In all of these relations it is assumed that the left-hand sides make sense. For instance, in (D3) one needs $a, b, c \in R$ with $1 + abc \in R^*$. $D$ is a functor from (commutative) rings to abelian groups. It commutes with finite direct products.

2.2. Put $K_2(n, R) = \ker(\mathrm{St}(n, R) \longrightarrow \mathrm{SL}(n, R))$, so that $K_2(R) = \varinjlim K_2(n, R)$. Put $K_2(\infty, R) = K_2(R)$. Relations (D1), (D2), (D3) imply the relations in [3, §9] and vice versa. So the rule

$$\langle a, b\rangle \longmapsto x_{21}\left(\frac{-b}{1+ab}\right)x_{12}(a)x_{21}(b)x_{12}\left(\frac{-a}{1+ab}\right)h_{12}^{-1}(1+ab)$$

defines a homomorphism $D(R) \longrightarrow K_2(R)$ factoring through $K_2(3, R)$.

2.3. DEFINITION. $R$ is called *3-fold stable* if, for any triple of unimodular sequences $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ there exists $r \in R$ such that $a_i + b_i r \in R^*$ for $i = 1, 2, 3$. (Recall that $(a, b)$ is called unimodular if $aR + bR = R$.) Similar definitions can be given for $k$-fold stability, e.g., 1-fold stability is the strongest of Bass' stable range conditions $SR_n(R)$ [1]. The condition of 3-fold stability is

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