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CONCAVITY OF MAGNETIZATION FOR A CLASS OF EVEN FERROMAGNETS[1]

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1. **Introduction.** Let E be the set of even probability measures which satisfy $\int \exp(kx^2)\rho(dx) < \infty$ for all $k \geq 0$ sufficiently small. Given an integer $N \geq 1$, real numbers $h \geq 0$ and $J_{ij} \geq 0$, $1 \leq i \leq j \leq N$, and measures $\rho_i \in E$, $1 \leq i \leq N$, we define [11, p. 273] real-valued random variables X_i, $1 \leq i \leq N$, with the joint distribution

$$\tau_h(dx_1, \ldots, dx_N) = \frac{\exp(\Sigma_{1 \leq i < j \leq N} J_{ij} x_i x_j + h \Sigma_{1 \leq i \leq N} x_i) \rho_1(dx_1) \ldots \rho_N(dx_N)}{Z(h)}. \tag{1}$$

$Z(h)$, the partition function, is given by the formula

$$Z(h) = \int_{R^N} \cdots \int \exp\left(\sum J_{ij} x_i x_j + h \sum x_i\right) \rho_1(dx_1) \cdots \rho_N(dx_N). \tag{2}$$

The J_{ij} are assumed to be so small that the integral in (2) converges for all $h \geq 0$. The inequalities we discuss are to hold for all $h \geq 0$ and all $J_{ij} \geq 0$ subject only to this restriction. The choice of ρ_i as the Bernoulli measure $b(dx) = \frac{1}{2}(\delta(x - 1) + \delta(x + 1))$ gives the classical Ising model.

We define the average magnetization per site, $m(h)$, by the formula

$$m(h) = \frac{1}{N} \frac{d}{dh} \ln Z(h) = \frac{1}{N} \sum_{i=1}^{N} E\{X_i\} \tag{3}$$

and consider inequalities on $m(h)$ and its derivatives. While the inequalities $m(h) \geq 0$, $dm(h)/dh \geq 0$ hold for any $\rho_i \in E$ [7, pp. 76–77], the concavity of $m(h)$, i.e.

$$d^2 m(h)/dh^2 \leq 0, \tag{4}$$

requires that further restrictions be placed on the ρ_i. Essentially, (4) is known to hold only in the Ising case and in models which can be built out of Ising models in a suitable way [4], [6]. Measures for which (4) fails are known [6].

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