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ERGODIC EQUIVALENCE RELATIONS, COHOMOLOGY, AND VON NEUMANN ALGEBRAS

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1. **Introduction.** Throughout, $(X, \mathcal{B})$ will be a standard Borel space, G some countable group of automorphisms, R_G the equivalence relation $\{(x, g \cdot x), g \in G\}$, and μ a σ-finite measure on X. For μ quasi-invariant, the orbit structure of the action has been studied by Dye [4], [5], Krieger [8]–[13], and others. Here, ignoring G and focusing on R_G via an axiomatization, and studying a cohomology for R_G, we obtain a variety of results about group actions and von Neumann algebras. The major results are stated below.

2. **Equivalence relations.** R will be an equivalence relation on X with all equivalence classes countable, and $R \in \mathcal{B} \times \mathcal{B}$.

THEOREM 1. *Every R is an R_G.*

Properties of G-actions translate into properties of R_G which can be stated with no G in sight, e.g., quasi-invariance, ergodicity. Let μ be quasi-invariant, and let $\mathcal{C} = \mathcal{B} \times \mathcal{B}|_R$ and $P_l(x, y) = x$, $P_r(x, y) = y$. Now $\mathcal{C}$ has a natural measure class as follows:

THEOREM 2. *The formula $\nu_l(C) = \int |P_l^{-1}(x) \cap C| d\mu(x)$, where $|\cdot|$ is cardinality, and a similar formula for ν_r define equivalent σ-finite measures on $\mathcal{C}$.*

The Radon-Nikodym derivative is the function $D = d\nu_r/d\nu_l$ on R. If $R = R_G$, then $d(\mu \cdot g)/d\mu(x) = D(x, gx)$. Moreover, D is a cocycle in that $D(x, y)D(y, z) = D(x, z)$ a.e. and the D' arising from a μ' equivalent to μ is cohomologous to D.

For ergodic R, one has a classification into types which are I_n, $n = 1, \ldots, \infty$, II_1, II_∞ and III as in [3]. For $j = 1, 2$, relations R_j on $(X_j, \mathcal{B}_j, \mu_j)$ are isomorphic if there is a Borel isomorphism a: $X_1 \to X_2$ with $\mu \sim \mu \circ a^{-1}$ and $R_2(a(x)) = a(R_1(x))$ a.e. If the R_j are ergodic, they are principal groupoids and, hence, define virtual groups [14].

THEOREM 3. *R_1 and R_2 define isomorphic virtual groups iff each is isomorphic to a restriction of the other, where the restriction of R to H is $R \cap H \times H$. Hence, the two notions of isomorphism coincide if R_1 and R_2 are both of infinite type.*

Hyperfiniteness in terms of R becomes: $\exists\, R_n \uparrow R$ with $|R_n(x)|$ finite $\forall n$, $\forall x$.

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