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COUNTING THE FACES OF CUT-UP SPACES

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Let us take a topological space X and "cut" it along certain subspaces $H_1, \ldots, H_n$. The connected components that remain we call the *regions* of the *topological dissection* of X by the *cut spaces* $H_1, \ldots, H_n$.

How many regions are there? There is a large literature on this question for particular kinds of dissections. For instance much of the study of arrangements of hyperplanes in Euclidean and projective spaces has been concerned with counting regions, bounded regions, and lower-dimensional faces. As another example: into how many pieces is the plane cut by a given set of lines or curves? Answers have been found mainly for straight cuts in low dimensions or general position, some notable exceptions being Steiner's interest in circular and spherical cuts and Winder's and our work on arbitrary arrangements of hyperplanes. (For references see [1, Chapter 18], [2], [5], [6].)

What has been lacking is a unified theory and the power to handle a great variety of dissections. In a forthcoming paper [6] we hope to provide such a theory. Here we shall mention most of the principal results and some areas of application.

1. **The fundamental relations.** The main idea of our theory is to count regions in terms of the set L of intersections of the cuts or the set L^c of connected components of intersections (we let X be in both sets but $\varnothing$ be in neither). These sets are ordered by inclusion. Thus L (and similarly L^c) has a *Möbius function* (cf. [3]), defined recursively by

$$\mu(S, T) = \delta(S, T) - \sum \{\mu(S, U) : S \subseteq U \subsetneq T\}, \qquad (S, T) \in L^2.$$

Here δ is the Kronecker delta. We also need the *combinatorial Euler number* of a space X, defined by $\kappa(X) = \chi - 1$, where χ means the Euler characteristic of the one-point compactification of X. If X is partitioned into n_k open k-cells ($k = 0, 1, \ldots, d$), then $\kappa(X) = n_0 - n_1 + \cdots \pm n_d$.

A dissection H of X induces a dissection of $T \in L$ by $H_T = \{H \cap T : \varnothing \neq H \cap T \neq T\}$. The *faces* of H are the regions of all the H_T for $T \in L$.

THEOREM 1. *Suppose X is dissected by $H_1, \ldots, H_n$ into regions $R_1, \ldots, R_m$ ($m \geqslant 0$), and every face is a disjoint, finite union of open topological cells.*

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