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IMAGES OF HOMOGENEOUS VECTOR BUNDLES AND VARIETIES OF COMPLEXES

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Let G be a connected algebraic group with a given representation on a vector space V. Let W be a subspace of V. I propose to study the union of all the translates of W by G, $G \cdot W$.

Let P be a subgroup of G that stabilizes W. Let $X \longrightarrow G/P$ be the homogeneous vector bundle over G/P, associated to the representation of P on W. Explicitly

$$X = \{(g, w) \in G \times W \text{ modulo } (g, w) \sim (gp^{-1}, pw) \text{ for } p \in P\}.$$

The representation $G \times V \longrightarrow V$ induces a morphism $f\colon X \longrightarrow V$. The image of f is $G \cdot W$.

THEOREM. *Assume G/P is complete. Then $G \cdot W$ is a closed subvariety of V. Furthermore, if the characteristic of the ground field is zero, and if the actions of G on V and of P on W are completely reducible, then $G \cdot W$ is a normal Cohen-Macaulay variety, and if f is birational, then $G \cdot W$ has rational singularities.*

The proof of this theorem uses the Borel-Weil-Bott theorem on the cohomology of homogeneous vector bundles [1] together with some facts surrounding the theory of rational resolutions [5].

The application that I have in mind for this theorem is the study of the singularities of the varieties of complexes introduced by Buchsbaum and Eisenbud [2].

I will first state what these varieties are. Let $K^0, \ldots, K^n$ be a sequence of vector spaces. Let V be the direct sum of $\operatorname{Hom}(K^0, K^1), \ldots, \operatorname{Hom}(K^{n-1}, K^n)$. A point a in V is denoted $(a_1, \ldots, a_n)$, where $a_i \in \operatorname{Hom}(K^{i-1}, K^i)$. A point a in V represents a complex if $a_{i+1} \circ a_i = 0$ for $0 < i < n$. The rank of a is the sequence of integers, $(\operatorname{rank} a_1, \ldots, \operatorname{rank} a_n)$, where rank b is the dimension of the image of the homomorphism b. If $(m_1, \ldots, m_n)$ is the rank of a complex, then $m_1 \leqslant \dim K^0$, $m_n \leqslant \dim K^n$, and $m_i + m_{i+1} \leqslant \dim K^i$ for $0 < i < n$. Conversely, any such sequence is the rank of a complex. Let M be the set of such sequences.

If $m \in M$, define the variety of Buchsbaum and Eisenbud, B-E(m), to be the variety of complexes a, such that $\operatorname{rank} a_i \leqslant m_i$ for $1 \leqslant i \leqslant n$.

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