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# THE INVALIDITY OF THE CALDERON-ZYGMUND INEQUALITY FOR SINGULAR INTEGRALS OVER LOCAL FIELDS

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We will show that the Calderón-Zygmund inequality, $\|T_\omega\|_p \leqslant C(p, r)\|\omega\|_r$, is not valid in the local field setting. A complete proof of the validity of this inequality in the case of singular integrals over $\mathbf{R}^n$ can be found in Dunford and Schwartz, *Linear operators*, Vol. 2. We use the theory of regular functions as developed by M. Taibleson [4] and the F. and M. Riesz theorem for local fields as proved by J. Chao [1].

We assume the reader is familiar with elementary local field analysis and singular integrals in general. In the following work $K$ will denote a local field (non-discrete, zero-dimensional, locally compact field), $B^n = \{x \in K: |x| \leqslant q^{-n}\}$, $D^n = \{x \in K: |x| = q^{-n}\}$, and $\xi_A$ the characteristic function of the set $A$. Haar measure $\lambda$ is normalized so that $\lambda(B^0) = 1$ $(\lambda(B^1) = q^{-1})$ and the prime $\pi$ is chosen so that $\pi B^0 = B^1$. The fundamental character $\chi$ is trivial on $B^0$ and nontrivial on $B^{-1}$. $C_{00}$ and $C_0$ denote the continuous functions with compact support and the continuous functions that vanish at infinity, respectively.

DEFINITION. For $x \in k$, $k \in \mathbf{Z}$, let

$$f(x, -k) = \begin{cases} 0, & k < 2, \\ \xi_{D^0}(x) \sum_{j=2}^{k} \chi(\pi^{-j}x) & \text{if } k \geqslant 2. \end{cases}$$

LEMMA 1. *The function $f$ defined above is regular.*

PROOF. A function $g$: $K \times \mathbf{Z} \longrightarrow \mathbf{C}$ is said to be regular if

$$g(x, k) = q^{-k} \int_{B^{-k}} g(y - x, k - 1)\, dy.$$

A straightforward calculation shows that $f$ satisfies this equality. □

LEMMA 2.

(a) $$\hat{f}(x, -k) = \frac{q-1}{q} \sum_{j=2}^{k} \xi_{\pi^{-j}+B^0}(x) - \frac{1}{q} \sum_{j=2}^{k} \xi_{\pi^{-j}+D^{-1}}(x).$$

(b) $\|f(\cdot, -k)\|_2 = \{(q-1)(k-1)/q\}^{1/2}$ *for* $k \geqslant 2$.

(c) $\|f(\cdot, -k)\|_r \leqslant \{(q-1)(k-1)/q\}^{(r-1)/r}$ *for* $k \geqslant 2$, $\quad 2 < r < \infty$.

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