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ON THE GEOMETRY OF NONCOMMUTATIVE SPECTRAL THEORY

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We shall consider an order unit space (A, e) and a base-norm space (V, K) in separating order and norm duality with A pointwise monotone σ-complete, i.e. for every descending sequence $\{a_n\}$ in A^+ there exists $a \in A$ such that $\langle a, x\rangle = \lim_n \langle a_n, x\rangle$ for $x \in K$. (See [1] for definitions and proofs.) We write $M \in \mathcal{T}_A$ if M is a weakly closed supporting hyperplane of A^+ and $\widetilde{F} = \bigcap \{M \in \mathcal{T}_A \mid F \subset M\}$ for $F \subset A^+$. (One may think of $\widetilde{F}$ as a "minimal tangent space" for A^+ at F.) M is a *smooth order ideal* of A if $M = (A^+ \cap M)^\sim$, and F is a *semiexposed face* of A^+ if $F = A^+ \cap \widetilde{F}$. For a projection P: $A \rightarrow A$ we write $\mathrm{im}^+ P = A^+ \cap \mathrm{im}\, P$, $\ker^+ P = A^+ \cap \ker P$. Two projections P, Q: $A \rightarrow A$ are *quasi-complementary* (q.c) if $\mathrm{im}^+ Q = \ker^+ P$, $\ker^+ Q = \mathrm{im}^+ P$. Similar definitions apply with V in place of A. A weakly continuous positive projection P of A (or V) with $\|P\| \leqslant 1$ is *smooth* if $\ker P$ is a smooth order ideal. A projection R of V is *neutral* if $\|Rv\| = \|v\|$ implies $Rv = v$ for $v \in V^+$. This term relates to physical filters which are "neutral" in that when a beam passes through with intensity undiminished ($\|Rv\| = \|v\|$), then the filter is neutral to it ($Rv = v$).

THEOREM 1. *For projections on A the following are equivalent*: (i) *P, Q are q.c. and so are the dual projections P^*, Q^**; (ii) *P, Q are q.c. and both are smooth*; (iii) *P^*, Q^* are q.c. and both are smooth*; (iv) *P, Q are q.c. and P^*, Q^* are neutral.*

P: $A \rightarrow A$ is a *P-projection* (in symbols $P \in \mathcal{P}$) if it admits a (necessarily unique) q.c. $P' = Q$ such that (i)–(iv) hold. To $P \in \mathcal{P}$ we associate a *projective unit* $u = Pe \in A$ ($u \in \mathcal{U}$) and a *projective face* $F = K \cap \mathrm{im}\, P^*$ of K ($F \in \mathcal{F}$). We write $F_P = K \cap \mathrm{im}\, P^*$ and $F_P^\# = F_{P'}$. Now $\mathcal{P}$, $\mathcal{U}$, $\mathcal{F}$ *are in natural* 1-1 *correspondence.* $P \in \mathcal{P}$ is *compatible with* $a \in A$ if $a = Pa + P'a$; when $a = Qe$ with $Q \in \mathcal{P}$, this will hold iff $P, Q \in \mathcal{P}$ commute, then we say *P, Q are compatible.* Also we say $P \in \mathcal{P}$ is *bicompatible* with $a \in A$ if P is compatible with a and with all $Q \in \mathcal{P}$ compatible with a. An affine retraction ρ: $K \rightarrow K' \subset K$ is said to be *transversal* at $F \subset K'$ if $\rho(y) = \rho(z)$ implies $y - z \in \widetilde{F}$.

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