

Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are Banach spaces, \{P_\tau\} and \{Q_\tau\} are families of projection operators on $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively which converge strongly as $\tau \to \infty$ to the respective identity operators, and $A$ is a bounded linear transformation from $\mathcal{B}_1$ to $\mathcal{B}_2$. One says that the projection method $(P_\tau, Q_\tau)$ is applicable to $A$ if, roughly speaking, $(Q_\tau A P_\tau)^{-1}$ converges strongly to $A^{-1}$ as $\tau \to \infty$. More precisely what is required is that $Q_\tau A P_\tau$ be invertible for sufficiently large $\tau$ and that $(Q_\tau A P_\tau)^{-1} Q_\tau$ converge strongly as $\tau \to \infty$. (Then $A$ is necessarily invertible and the strong limit is $A^{-1}$.)

To give an example, the prototype of those considered in this book, let $a$ be a bounded function defined on the unit circle having Fourier coefficients $a_k (k=0, \pm 1, \cdots)$, and consider the operator $A$ on $l_2$ of the positive integers defined by

$$ A\{\xi_j\} = \left\{ \sum_{k=1}^{\infty} a_{j-k} \xi_k \right\}_{j=1}^{\infty}. $$

This is the (semi-infinite) Toeplitz operator associated with $a$. The projections are the simplest ones: $P_n = Q_n =$ projection on the subspace of sequences \{\xi_j\} satisfying $\xi_j = 0$ for $j > n$. The operator $P_n A P_n$ may then be represented by the finite Toeplitz matrix

$$ A_n = (a_{j-k})_{j,k=1}^{n} $$

and the question is whether these matrices are invertible from some $n$ onward and, if so, whether the inverses of these matrices converge strongly