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DERIVATIVES OF ENTIRE FUNCTIONS AND A QUESTION OF PÓLYA

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The purpose of this note is to announce a partial solution to an old question of Pólya.

To state concisely his question and our results we introduce the following notation: For each integer $p \geqslant 0$, denote by V_{2p} the class of entire functions of the form $f(z) = \exp(-az^{2p+2})g(z)$ where $a \geqslant 0$ and $g(z)$ is a constant multiple of a real entire function of genus $\leqslant 2p + 1$ with only real zeros. Recall that a real entire function is one which assumes real values on the real axis. Now set $U_0 = V_0$, and for $p \geqslant 1$, set $U_{2p} = V_{2p} - V_{2p-2}$.[2]

The class U_0, often called the Laguerre-Pólya class, is of particular interest, for a classical theorem of Laguerre [4] and Pólya [6] asserts that $f \in U_0$ if and only if it can be uniformly approximated on discs about the origin by a sequence of polynomials with only real zeros. Consequently, if $f \in U_0$, then $f^{(n)} \in U_0$, $n = 1, 2, \cdots$; in particular, $f \in U_0$ implies $f^{(n)}$ has only real zeros $n = 1, 2, \cdots$.

In 1914, Pólya [7] asked whether the converse is true: *If an entire function f and all its derivatives have only real zeros, is $f \in U_0$?*

Pólya showed [7], [8] that if $f = Pe^Q$ where P and Q are polynomials, then, except for functions of the form ae^{bz} (a and b constants, b complex), the answer is yes. Moreover, in [8] he conjectured that, in the general case, the *only* exceptions are functions f of the form $f(z) = ae^{bz}$ or $f(z) = a(e^{icz} - e^{id})$ where a, b, c, and d are constants, c and d real, b complex. In [1] and [2], M. Ålander proved that the answer to Pólya's question is affirmative for all $f \in U_{2p}$ with $p \leqslant 2$ and in [3] purported to have extended this result to arbitrary p. However, in a famous survey article on zeros of successive derivatives [9], Pólya refers to Ålander's papers [1] and [2] but not to his more general result [3]. The first author of this announcement while a

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[2]This classification was introducted by Ålander [3].