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BOOK REVIEW

Conjugacy classes in algebraic groups, by Robert Steinberg, Springer-Verlag, New York, 1974, vi+159 pp., $7.00

Suppose A and B are invertible $n\times n$ complex matrices. How can one tell if A and B are conjugate—that is, if there is an invertible $n\times n$ complex matrix C such that $B=CAC^{-1}$? The solution, of course, is to examine the Jordan canonical form of the two matrices. But now suppose A and B lie in some algebraic subgroup of the group of all invertible $n\times n$ complex matrices—that is, in a subgroup determined by some polynomial conditions on the matrix entries, for example the matrices of determinant one. Now how can one tell if A and B are conjugate in that subgroup?

In more formal language, suppose G is a linear algebraic group over an algebraically closed field k, and let G act on itself by conjugation. The quotient set V of G under this action is the set of conjugacy classes of G. What one really wants to know is the structure of V, or of some subsets of it determined by looking at conjugacy classes of elements of special types. It is to the analysis of this structure that Robert Steinberg's lecture notes *Conjugacy classes in algebraic groups* are directed.

To begin, it is possible to carry out some of the Jordan decomposition in G. Every element x of G, can be written as a product $x_s x_u$, where x_s acts semisimply on any finite dimensional vector space V on which G acts linearly and algebraically; that is, V has a basis of eigenvalues of x_s, while x_u acts unipotently on V; that is all its eigenvalues on V are one. Call x semisimple or unipotent if $x=x_s$ or $x=x_u$. The first major result deals with the description of the conjugacy classes of semisimple elements.

An algebraic group is semisimple if it has no nontrivial normal, solvable algebraic subgroup, and it is simply connected if it has no nontrivial central extensions with finite kernel. Suppose G is such a group. There is a special distinguished set $V_1, \cdots, V_n$ of finite dimensional vector spaces with G-action (in technical language, the V_i are the representations corresponding to the fundamental weights of G for some choice of root system). Consider the functions X_i, $i=1, 2, \cdots, n$, on G where $X_i(x)$ is the trace of the matrix of the action of x on V_i, and consider the map $p:G\to k^{(n)}$ defined by $p(x)=(X_1(x), \cdots, X_n(x))$. Then the result is that p induces a bijection between the set of conjugacy classes of semisimple elements of G and $k^{(n)}$.