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EXISTENCE AND REGULARITY ALMOST EVERYWHERE OF SOLUTIONS TO ELLIPTIC VARIATIONAL PROBLEMS WITH CONSTRAINTS

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This is a research announcement of results [A1] the full details and proofs of which have been submitted for publication elsewhere. We study the structure of m dimensional subsets of $\mathbf{R}^n$ which are well behaved with respect to deformations of $\mathbf{R}^n$ and also show the existence of such sets as solutions to geometric variational problems satisfying various constraints.

Suppose, for example, one is given several positive numbers $a_1, a_2, \cdots, a_N$ and is asked to find disjointed regions $A_1, A_2, \cdots, A_N$ in $\mathbf{R}^n$ such that A_i has volume a_i for each i and the $n-1$ dimensional area of $S = \bigcup\{\text{Boundary}(A_i): i = 1, \cdots, N\}$ is as small as possible. For $n = 3$ this is a common formulation of a variational problem associated with compound soap bubbles. As a variant of this problem one could set $A_0 = \mathbf{R}^n \sim \bigcup_i \text{Closure}(A_i)$ and attempt to minimize the sum of the weighted areas of the various interfaces $\{\text{Boundary}(A_i) \cap \text{Boundary}(A_j)\}_{i,j}$, or perhaps the weighted integrals over these interfaces of various geometric integrands. For $n = 2, 3$ such minimal partitioning hypersurfaces have been the subject of numerous papers in mathematics, physics, and especially biology for the past several centuries (see, for example, [TD, Chapter 4, pp. 88–125] for a discussion and references). Among other things we give the first mathematical proof of the general existence of such surfaces. The methods are representative of those required to show the existence and regularity of solutions to a variety of geometric variational problems with constraints; e.g. capillarity problems, minimal surfaces avoiding obstacles, variational problems with partially free boundaries, etc.

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