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SUMS OF kTH POWERS IN THE RING OF POLYNOMIALS WITH INTEGER COEFFICIENTS

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Suppose R is a ring with identity element and k is a positive integer. Let $J(k, R)$ denote the subring of R generated by its kth powers. If Z denotes the ring of integers, then $G(k, R) = \{a \in Z: aR \subset J(k, R)\}$ is an ideal of Z.

Let $Z[x]$ denote the ring of polynomials over Z and suppose $a \in R$. Since the map $p(x) \longrightarrow p(a)$ is a homomorphism of $Z[x]$ into R, the well-known identity (see [3, p. 325])

$$k!x = \sum_{i=0}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} \{(x+i)^k - i^k\} \tag{1}$$

in $Z[x]$ tells us that $k! \in G(k, Z[x]) \subseteq G(k, R)$. Since Z is a cyclic group under addition, this shows that $G(k, R)$ is generated by its minimal positive element, which we denote by $m(k, R)$. Abbreviating $m(k, Z[x])$ by $m(k)$, we then have $m(k, R)|m(k)$ and $m(k)|k!$.

Thus $m(k)$ is the smallest positive integer a for which there is an identity of the form

$$ax = \sum_{i=1}^{n} a_i[g_i(x)]^k \tag{2}$$

where $a_1, \cdots, a_n \in Z$ and $g_1(x), \cdots, g_n(x) \in Z[x]$.

On differentiating (2) with respect to x we have $k|m(k)$. Thus if R is any ring with identity,

$$k|m(k), \quad m(k, R)|m(k), \quad \text{and} \quad m(k)|k!. \tag{3}$$

For any $k \geqslant 1$ in Z, let $P_1(k)$ denote the set of primes less than k that divide k, and let $P_2(k)$ denote the set of primes less than k that fail to divide k. If p is a prime and $r \geqslant 1$, $m > 1$ are integers, then a number

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