BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 6, November 1974

ON THE DETERMINATION OF A HILL'S EQUATION FROM ITS SPECTRUM

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Communicated by Cathleen Morawetz, March 26, 1974

A Hill's equation is an equation of the form:

$$y'' + [\lambda - q(z)]y = 0, \qquad q(z + \pi) = q(z), \tag{1}$$

where $q(z)$ is assumed to be integrable over $[0, \pi]$. Without loss of generality, it is customary to assume that $\int_0^\pi q(z)\,dz = 0$. The discriminant of (1) is defined by

$$\Delta(\lambda) = y_1(\pi) + y_2'(\pi),$$

where y_1 and y_2 are solutions of (1) satisfying $y_1(0) = y_2'(0) = 1$ and $y_1'(0) = y_2(0) = 0$.

The set of values of λ for which $|\Delta| > 2$ consists of a finite or an infinite number of finite disjoint intervals and one infinite interval. These intervals are called instability intervals, since (1) has no solution which is bounded for all real z in these intervals. When $|\Delta| < 2$, all solutions of (1) are bounded for all real z and the corresponding intervals are called stability intervals. Pertinent information about stability and instability intervals of (1) can be found in Magnus and Winkler **[1]**.

The following result has been proved:

THEOREM. *If $q(z)$ is real and integrable, and if precisely n finite instability intervals fail to vanish, then $q(z)$ must satisfy a differential equation of the form*

$$q^{(2n)} + H(q, q', \cdots, q^{(2n-2)}) = 0, \quad a.e. \tag{2}$$

where H is a polynomial of maximal degree $n+2$.

Borg **[2]**, Hochstadt **[3]** and Ungar **[4]** proved this theorem for the case $n=0$, i.e. when all finite instability intervals vanish, and found that

$$q(z) = 0, \quad \text{a.e.} \tag{3}$$

For the case $n=1$, Hochstadt **[3]** showed that $q(z)$ is the elliptic function which satisfies

$$q'' = 3q^2 + Aq + B, \quad \text{a.e.} \tag{4}$$

AMS (MOS) subject classifications (1970). Primary 34B30; Secondary 34E05.

[1] I thank my advisor, Harry Hochstadt, for his generous advice and guidance.