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NONLINEAR PARABOLIC EQUATIONS AND PROBABILITY

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Introduction. The linear parabolic differential equation

$$\frac{\partial u(t, x)}{\partial t} = \sum_{ij} a_{ij}(t, x) \frac{\partial^2 u(t, x)}{\partial x_i \, \partial x_j} + \sum_i b_i(t, x) \frac{\partial u(t, x)}{\partial x_i} \tag{1}$$

and its connection with Markov processes with continuous paths, called diffusion processes, has been studied extensively, for example in the books of Doob [3], Itô and McKean [8], Dynkin [2], Mandl [12], Gihman and Skorohod [5], and in the papers of Stroock and Varadhan [15]. The equation (1), called the diffusion equation, is associated with a family $P(t, x)$ of probability measures on the space $C([0, \infty), R^m)$ of continuous R^m-valued functions on $[0, \infty)$. Each of these measures defines a Markov process $x(s, w)$ with continuous trajectories starting at the point x at time t, and the solution $u(t, x)$ of (1) may be represented on an interval $[0, T]$ in terms of its initial value $u(0, x)$ by the formula

$$u(t, x) = \int u(0, x(T, w))P(T - t, x)\,(dw). \tag{2}$$

Note that the process $x(s, w)$ is scaled in reverse, i.e. $s = T - t$ for $t \leqq T$.

A common example of a diffusion equation is the Fokker-Planck equation of statistical mechanics for which the solution $u(t, x)$ represents the density in phase space at time t for a fluid particle. This equation is usually derived as a nonlinear equation with coefficients a_{ij} and b_i dependent in some way upon the solution u (see [16] for several such derivations). In textbooks, the dependence upon u is neglected to simplify the theory. One approach to the study of the nonlinear equation was introduced by McKean [13], [14] based upon a derivation of Kac [9] which gives coefficients as a function of the value of u. Under certain smoothness conditions for the coefficients, McKean showed the existence and uniqueness of a

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