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P_n-SPACES AND n-FOLD LOOP SPACES

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The purpose of this paper is to present a characterization of n-fold loop spaces for $1 \leqq n < \infty$. The approach is in the same spirit as G. Segal's investigation of infinite loop spaces via "special Γ-spaces" [4]. Category theoretic terminology not explained here may be found in [1].

I. The P-construction on small pointed categories. Let P_1 be the category with objects the finite ordered sets, $\boldsymbol{n}=\{0, \cdots, n\}$, and with morphism sets $P_1(\boldsymbol{n}, \boldsymbol{m})=\{f:\boldsymbol{n}\to\boldsymbol{m} \mid f(0)=0; f(i)\leqq f(j) \text{ if } i<j \text{ and } f(j)\neq 0\}$. Let $\#:P_1\times P_1\to P_1$ be the bifunctor such that $\boldsymbol{n}\#\boldsymbol{m}=\{0, \cdots, n+m\}$ and such that if $f_i \in P_1(\boldsymbol{n}_i, \boldsymbol{m}_i)$ for $i=1, 2$,

$$\begin{aligned} f_1\#f_2(j) &= f_1(j), && 0 \leqq j \leqq n_1; \\ &= f_2(j-n_1)+m_1, && n_1 < j \leqq n_1+n_2 \text{ and } f_2(j-n_1)\neq 0; \\ &= 0, && n_1 < j \leqq n_1+n_2 \text{ and } f_2(j-n_1) = 0. \end{aligned}$$

Then $\#$ is strictly associative and $\boldsymbol{0}$ is a two-sided unit for $\#$ and a unique null-object for P_1.

Let C be a small category with a unique null-object e. For each $a \in C$, we will denote by N_a and O_a the unique morphisms in $C(a, e)$ and $C(e, a)$ respectively. We now construct a strictly monoidal category $P(C)$, which one might describe as a "wreath-product" of P_1 with C.

The objects of $P(C)$ are the finite sequences, $\langle a_1, \cdots, a_n\rangle$, of nonnull objects of C (including the empty sequence $\langle\ \rangle$). If $\alpha=\langle a_1, \cdots, a_n\rangle$ and $\beta=\langle b_1, \cdots, b_k\rangle$, we set

$$P(C)(\alpha, \beta) = \{(f; h_1, \cdots, h_n) \mid f \in P_1(\boldsymbol{n}, \boldsymbol{k}), h_i \in C(a_i, b_{f(i)})\}.$$

(By convention, $b_0=e$.) Composition of morphisms is defined according to the rule:

$$(f'; h'_1, \cdots, h'_k)(f; h_1, \cdots, h_n) = (f'f; h'_{f(1)}h_1, \cdots, h'_{f(n)}h_n).$$

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