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ADDITIVE COMMUTATORS BETWEEN 2×2 INTEGRAL MATRIX REPRESENTATIONS OF ORDERS IN IDENTICAL OR DIFFERENT QUADRATIC NUMBER FIELDS

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The following theorem holds:

THEOREM 1. *Let A, B be two integral 2×2 matrices. Let the characteristic roots of A be α, α' and let the characteristic roots of B be β, β', all assumed irrational. Then the determinant of*

$$(*)\qquad L = AB - BA$$

is a negative norm in both $Q(\alpha)$, $Q(\beta)$.

REMARK. The proof of this theorem gives an algorithmic procedure for expressing an integer as a norm in a quadratic field.

PROOF. There exists[2] an integral matrix S with the property that $S^{-1}AS$ is the companion matrix

$$\begin{pmatrix} 0 & 1 \\ -\det A & \operatorname{tr} A \end{pmatrix}$$

of A. Since the companion matrix has the characteristic vectors $(1, \alpha)'$, $(1, \alpha')'$ the matrix $T=\left(\begin{smallmatrix} 1 & 1 \\ \alpha & \alpha' \end{smallmatrix}\right)$ has the property that $T^{-1}S^{-1}AST=\left(\begin{smallmatrix} \alpha & \\ & \alpha' \end{smallmatrix}\right)$. Apply then the same similarity also to B and to L, i.e. to $(*)$. Let the outcome of this be denoted by

$$(**)\qquad \begin{pmatrix} \alpha & \\ & \alpha' \end{pmatrix} B^{(\alpha)} - B^{(\alpha)} \begin{pmatrix} \alpha & \\ & \alpha' \end{pmatrix} = L^{(\alpha)} = \begin{pmatrix} 0 & l_2 \\ l_3 & 0 \end{pmatrix};$$

then l_2, l_3 are elements in $Q(\alpha)$.

Apply the similarity defined by T^{-1} to $L^{(\alpha)}$. The result must be rational. A straightforward computation using the fact that $\alpha, \alpha' = -\frac{1}{2}(\operatorname{tr} A \pm \sqrt{m})$, with $m=(\operatorname{tr} A^2 - 4 \det A)$, shows that

$$\begin{pmatrix} 1 & 1 \\ \alpha & \alpha' \end{pmatrix} \begin{pmatrix} 0 & l_2 \\ l_3 & 0 \end{pmatrix} \begin{pmatrix} \alpha' & -1 \\ -\alpha & 1 \end{pmatrix} \frac{1}{\alpha' - \alpha} = -\frac{1}{\sqrt{m}} \begin{pmatrix} \alpha' l_3 - \alpha l_2 & l_2 - l_3 \\ \alpha'^2 l_3 - \alpha^2 l_2 & -\alpha' l_3 + \alpha l_2 \end{pmatrix}.$$

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[2] For further information in the number theoretic case on this see [1].