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ENUMERATION OF PAIRS OF PERMUTATIONS AND SEQUENCES[1]

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Let $\pi=(a_1, \cdots, a_n)$ denote a permutation of $Z_n=\{1, 2, \cdots, n\}$. A *rise* of π is a pair a_i, a_{i+1} with $a_i<a_{i+1}$; a *fall* is a pair a_i, a_{i+1} with $a_i>a_{i+1}$. Thus if $\rho=(b_1, \cdots, b_n)$ denotes another permutation of Z_n, the two pairs a_i, a_{i+1}; b_i, b_{i+1} are either both rises, both falls, a rise and a fall or a fall and a rise. We denote these four possibilities by *RR*, *FF*, *RF*, *FR*, respectively.

Let $\omega(n)$ denote the number of pairs of permutations π, ρ with *RR* forbidden. More generally let $\omega(n, k)$ denote the number of pairs π, ρ with exactly k occurrences of *RR*.

THEOREM 1. *We have*

$$\sum_{n=0}^{\infty}\omega(n)\frac{z^n}{n!\,n!}=\left\{\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{n!\,n!}\right\}^{-1}, \tag{1}$$

where $\omega(0)=\omega(1)=1$.

THEOREM 2.

$$\sum_{n=0}^{\infty}\frac{z^n}{n!\,n!}\sum_{k=0}^{n-1}\omega(n, k)x^k=\frac{1-x}{f(z(1-x))-x}, \tag{2}$$

where $f(z)=\sum_{n=0}^{\infty}(-1)^n(z^n/n!n!)$.

The pair π, ρ is said to be *amicable* if *RF* and *FR* are both forbidden. Let $\alpha(n)$ denote the number of amicable pairs of Z_n; more generally let $\alpha(n, k)$ denote the number of pairs π, ρ with k total occurrences of *RF* and *FR*.

THEOREM 3. *We have*

$$A(z)A(-z)=1, \tag{3}$$

where $A(z)=\sum_{n=0}^{\infty}\alpha(n)z^n/n!n!$.

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