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NONSMOOTHING OF ALGEBRAIC CYCLES ON GRASSMANN VARIETIES[1]

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1. **Introduction.** By a *cycle* Z of dimension r on a nonsingular algebraic variety X, we mean a formal linear combination $Z=\sum n_i Y_i$ of irreducible subvarieties Y_i of dimension r, with integer coefficients n_i. The smoothing problem for cycles asks whether a given cycle Z is equivalent (for a suitable equivalence relation of cycles, such as rational equivalence or algebraic equivalence) to a cycle $Z'=\sum n_i' Y_i'$, where the subvarieties Y_i' are all nonsingular. Let X be a nonsingular projective variety of dimension n over $\mathbf{C}$. Then for each cycle Z of dimension r on X we can assign a cohomology class $\delta(Z)\in H^{2n-2r}(X, \mathbf{Z})$. We say that two cycles Z, Z' are *homologically equivalent* if $\delta(Z)=\delta(Z')$. Our main result is that there are cycles on certain Grassmann varieties which cannot be smoothed, even for homological equivalence, which is weaker than rational or algebraic equivalence.

The smoothing problem was suggested by Borel and Haefliger [**2**, p. 497] in connection with their study of the cohomology class associated to a cycle. Hironaka [**7**] showed in characteristic zero that cycles of dimension $\leqq \min(3, \frac{1}{2}(n-1))$ can always be smoothed. On a nonsingular variety of any characteristic he showed that if $\dim Z \leqq \frac{1}{2}(n-1)$, then some multiple of Z can be smoothed. Kleiman [**8**] strengthened the latter result by showing that if $\dim Z<\frac{1}{2}(n+2)$, then $((q-1)!)Z$ can be smoothed, where $q=n-r$ is the codimension of Z. The specific cycle which we show to be nonsmoothable was suggested by Kleiman and Landolfi [**9**], who conjectured that it could not be smoothed.

Thom, in his famous paper [**11**], studied the closely related question of which homology classes on a smooth manifold can be represented as the homology class of a submanifold. He also answered negatively a question of Steenrod, which asked if every homology class on a manifold was the image by some continuous map of the fundamental class of another manifold. Note that in algebraic geometry if a homology class is the class of some cycle on a nonsingular variety X, then it is the image of the

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