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ON SEQUENCES OF MEASURES

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Dieudonné [2] has shown that a sequence (μ_n) of regular Borel measures on a compact space X converges weakly, i.e., on all bounded Borel functions, if only it converges on all open Baire sets. The result continues to hold if the μ_n are weakly compact linear maps from $C(X)$ to a locally convex vector space F. Such maps have an integral extension to all bounded Borel functions ϕ, and $\int \phi \, d\mu_n$ converges provided $\int_O d\mu_n$ converges for all open sets O [4], [5]. The Vitali-Hahn-Saks theorem is the set-function analogue of these results.

In this note the analogue of these results for sequences (μ_n) of measures with values in an arbitrary topological vector space F will be proved. In order to deal with set functions and linear maps at the same time, we work in the setting of Daniell-Stone, and consider linear maps $\mu:\mathscr{R}\to F$, where $\mathscr{R}$ is a vector lattice of real-valued functions on a set X closed under the Stone-operation $\phi\to\phi\wedge 1$, an "integration lattice" [1]. The examples we have in mind are (1) $\mathscr{R}=C^{00}(X)$, where X is locally compact, (2) $\mathscr{R}=\mathscr{E}(\mathscr{C})$, the step functions over a clan of sets on X, (3) $\mathscr{R}=c^{00}$, (4) $\mathscr{R}=l^\infty$. If an additive set function $\mu:\mathscr{C}\to F$ on the clan $\mathscr{C}$ is given, we extend it by linearity to $\mathscr{E}(\mathscr{C})$ and are in the present situation.

We denote by $\mathscr{O}_0^S$ the collection of sets in X whose indicator is majorized by a function in $\mathscr{R}$ and is the supremum of a sequence in $\mathscr{R}_+$. $\mathscr{O}_0^S$ consists of the open dominated $\mathscr{R}$-Baire sets [1]. We shall assume that every function in $\mathscr{R}$ is bounded and vanishes off some set in $\mathscr{O}_0^S$. Examples (1)–(4) have this property.

Then $\mathscr{R}$ is the union of the normed spaces $\mathscr{R}[O]=\{\phi\in\mathscr{R}:\phi=0 \text{ off } O\}$ under the supremum norm $\|\ \|_\infty$ and is given the inductive limit topology. X is given the initial uniformity and topology for the functions $\phi:X\to\bar{\mathbf{R}}$ $(\phi\in\mathscr{R})$, under which it is precompact. Its completion $\tilde{X}$ can be identified with the set of all Riesz-space characters $t:\mathscr{R}\to\mathbf{R}$ having $t(\phi\wedge 1)=t(\phi)\wedge 1$. Subtracting from $\tilde{X}$ the zero character, one obtains the locally compact *spectrum* $\hat{X}$ of $\mathscr{R}$. X is dense in $\hat{X}$, and the extensions $\hat{\phi}$ of $\phi\in\mathscr{R}$ to $\hat{X}$, the Gelfand transforms, are dense in $C^{00}(\hat{X})$. For the details see [1].

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