BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 5, September 1974

THE PERMANENT AT A MINIMUM ON CERTAIN CLASSES OF DOUBLY STOCHASTIC MATRICES[1]

BY MARK BLONDEAU HEDRICK

Communicated by Olga Taussky Todd, March 18, 1974

ABSTRACT. Both David London and Mark B. Hedrick have independently generalized a result of M. Marcus and M. Newman concerning the behavior of the permanent at a minimum on the set of doubly stochastic matrices. The author generalizes this last result and simplifies the proof appreciably. He proves the following. Let A be a doubly stochastic matrix and let X be a set of doubly stochastic matrices with the same (0, 1)-pattern as A in some neighborhood of A. If A is a critical point of the permanent relative to X, then per $A = \operatorname{per} A(i|j)$ for each positive a_{ij}.

In 1926 [**8**], B. L. van der Waerden conjectured that the permanent achieves a unique minimum on the set D_n of $n\times n$ doubly stochastic matrices at the matrix J_n (each of whose entries is $1/n$). The conjecture has some interesting interpretations in finite probability and finite combinatorics. However, it is only known to be true for n less than or equal to 5 [**6**], [**1**], [**2**] and for the class of positive semidefinite, hermitian matrices [**5**]. The most general result previously known was obtained independently by D. London [**4**] and by M. B. Hedrick [**3**] and stated that if A is a matrix in D_n at which the permanent achieves a local minimum relative to D_n, then per $A \leqq \operatorname{per} A(i|j)$ with equality for each positive a_{ij}. Both of these papers relied heavily on methods from the definitive work of M. Marcus and M. Newman [**6**] in which knowledge of the eigenvalues of AA^T was required. Since the relationship between the eigenvalues of A or AA^T and the permanent of A appears to be extremely nebulous [**7**], the author finds a great deal of beauty in the simplicity and purely combinatorial nature of the following proof.

THEOREM. *Let A be a doubly stochastic matrix, and let X be a set of doubly stochastic matrices with the same* (0, 1)*-pattern as A in some neighborhood of A. If A is a critical point of the permanent relative to X, then* per $A = \operatorname{per} A(i|j)$ *for each positive a_{ij}.*

AMS (MOS) subject classifications (1970). Primary 05B20, 60J05, 15A15, 15A51.

Key words and phrases. Permanent, doubly stochastic matrix, finite combinatorics, finite probability, critical point of the permanent relative to X.

[1] This paper is dedicated to Terrie.