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RESTRICTED APPROXIMATION AND INTERPOLATION

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There has been much interest recently in questions of approximating or interpolating continuous functions by polynomials which are subject to certain constraints. Among the problems considered are those of (a) monotone approximation [**5**], [**6**], [**10**], [**11**], (b) comonotone approximation [**8**], [**9**], (c) piecewise monotone interpolation [**3**], [**12**], [**13**], and (d) copositive approximation. We outline the problems below and state our results. The proofs will appear elsewhere.

(a) *Monotone approximation.* Let P_n be the set of all algebraic polynomials of degree $\leqq n$. Let $K=\{0\leqq k_1<k_2<\cdots<k_q\}$ be a set of integers, let $\varepsilon_i=\pm 1$, $i=1, 2, \cdots, q$, and let $f\in C^j[0, 1]$, where $j\geqq k_q$. Suppose that $\varepsilon_i f^{(k_i)}(x)\geqq 0$ for all $x\in[0, 1]$, $i=1, 2, \cdots, q$. (More generally, we may assume only that $\varepsilon_i\Delta^{(k_i)}f\geqq 0$, $i=1, 2, \cdots, q$, for nondifferentiable functions.) Let

$$E_n(f; K)=\inf\{\|f-p\|:p\in P_n, \varepsilon_i p^{(k_i)}(x)\geqq 0, x\in[0, 1], i=1, 2, \cdots, q\}.$$

$E_n(f; K)$ is called the *degree of monotone approximation* of f. Lorentz and Zeller [**6**] have shown that if $K=\{1\}$, and $f\in C[0, 1]$ is nondecreasing, then $E_n(f; K)=O(\omega(f; 1/n))$.

THEOREM 1. *Let $f\in C[0, 1]$ and assume that $\varepsilon_i\Delta^{(k_i)}f(x)>0$ in $[0, 1]$, $i=1, 2, \cdots, q$. Then $E_n(f; K)=O(\omega(f; 1/n))$.*

THEOREM 2. *Let $f\in C[0, 1]$, let k be an integer, and assume that $\Delta^{(k)}f(x)\geqq 0$ in $[0, 1]$. Then for any $\varepsilon>0$ there exists $d(k, \varepsilon)$ such that $E_n(f; K)\leqq d(k, \varepsilon)\omega(f; 1/n^{1-\varepsilon})$, for n sufficiently large.*

(b) *Comonotone approximation.* Let $f\in C[0, 1]$ be a function having a finite number of local extrema. Such a function is said to be *piecewise monotone*. The local extrema are called the *peaks* of f. Two functions f and g are said to be *comonotone* on $[0, 1]$ if f and g are increasing and decreasing on exactly the same subintervals of $[0, 1]$. Let f be piecewise monotone on $[0, 1]$ and let

$$E_n^*(f)=\inf\{\|f-p\|:p\in P_n, p \text{ comonotone with } f\}.$$

$E_n^*(f)$ is called the *degree of comonotone approximation* of f.

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