BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, July 1974

ZEROS OF FUNCTIONS IN THE BERGMAN SPACES

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Communicated by R. C. Buck, November 6, 1973

A function $f(z)$ analytic in the unit disk is said to belong to the Bergman space A^p $(0<p<\infty)$ if $\int_0^1\int_0^{2\pi}|f(re^{i\theta})|^p r\,dr\,d\theta<\infty$. It is clear that A^p contains the Hardy space H^p of analytic functions for which $\lim_{r\to 1}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta<\infty$. We adopt the convention that $A^\infty=H^\infty$, the space of bounded analytic functions in the disc.

Assuming that $f(0)\neq 0$, we list the zeros of f in order of nondecreasing modulus: $0<|z_1|\leqq|z_2|\leqq|z_3|\leqq\cdots<1$. We repeat z_i according to the multiplicity of the zero of f at z_i. The sequence $\{z_i\}$ is called the zero set of f. If $f\in A^p$ (resp. H^p), then z_i will be called an A^p (resp. H^p) zero set. It has long been known that H^p zero sets $(0<p\leqq\infty)$ are completely characterized by the condition $\prod_{k=1}^\infty 1/|z_k|<\infty$. (Equivalently, $\sum_{k=1}^\infty 1-|z_k|<\infty$.) In particular, the condition is independent of p. Our results show that the situation for A^p zero sets is considerably more complex.

LEMMA 1. *If $\{z_k\}$ is an A^p zero set $(0<p<\infty)$, then*

$$\prod_{K=1}^{N}\frac{1}{|z_k|}=O(N^{1/p}).$$

COROLLARY. *If $\{z_k\}$ is an A^p zero set $(0<p<\infty)$, then for each $\varepsilon>0$,*

$$\sum_{k=1}^{\infty}(1-|z_k|)\left\{\log\frac{1}{1-|z_k|}\right\}^{-1-\varepsilon}<\infty.$$

If $f(z)=\sum_{n=0}^\infty a_n z^n$, let $S_N^{(p)}=\sum_{k=1}^N |a_k|^p$, $p>0$.

LEMMA 2. *If $S_N^{(2)}=O(N^\alpha)$ for some $\alpha\geqq 1$, then $f\in A^p$ for all $p<2/\alpha$.*

LEMMA 3. *For some p, $1\leqq p\leqq 2$, suppose that $\sum_{N=1}^\infty N^{-p}S_N^{(p)}<\infty$ and $N^{1-p}S_N^{(p)}=O(1)$. Then $f\in A^{p'}$, $1/p+1/p'=1$.*

Lemma 1 is proved by an application of Jensen's theorem. Lemmas 2 and 3 follow from corresponding coefficient conditions, after a summation by parts. In particular, Lemma 3 is a consequence of the fact that

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AMS (MOS) subject classifications (1970). Primary 30A04; Secondary 30A78.
Key words and phrases. Bergman spaces, A^p spaces, A^p zero sets, H^p zero sets.