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# CARDINAL ADDITION AND THE AXIOM OF CHOICE

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Using the axiom of choice (AC) one can prove that cardinal arithmetic is trivial for infinite cardinals. That is:

$$m + n = \sup(m, n) = m \cdot n \quad \text{if one of } m, n \text{ are infinite.}$$

These two laws follow directly from the simpler looking laws:

(1) $m+m=m$ for all infinite cardinals $m$, and

(2) $m \cdot m=m$ for all infinite cardinals $m$.

In 1924 Tarski [**2**] gave a proof of (AC) from (2) and asked [**3**] if (AC) is provable from (1). We give a negative answer to this question by exhibiting a permutation model of ZFU (Zermelo-Fraenkel set theory modified so as to allow urelements) in which $C_2$ (the axiom of choice for families whose elements are pairs) is false but (1) is true.[1]

A permutation model (Specker [**1**]) is determined by a set of points[2] (i.e. urelements) $U$, a group $G$ of permutations of $U$, and a conjugated filter $J$ of subgroups of $G$.

Let $U$ be a countable set of points. $G$ and $J$ will be described in terms of a 1-1 correspondence between $U$ and $\omega^{(\omega)}=\{s:\omega\to\omega \mid (\exists n)(\forall j>n)s_j=0\}$. For notational simplicity, in the sequel we will identify $U$ with $\omega^{(\omega)}$. For $s \in U$, let the pseudo length of $s=\mu k$ $(\forall n>k, s_n=0)$. Call a set $A \subseteq U$ *bounded* if there is a finite bound on the pseudo lengths of elements of $A$. For any permutation $\varphi$ of $U$ call $\{a \mid \varphi(a)\neq a\}$ the *support of* $\varphi$. Let $G=\{\varphi \mid \varphi$ is a permutation of $U$ and the support of $\varphi$ is bounded$\}$. For

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[1] The proof of our result was completed near the end of December 1972. We learned subsequently that G. Sageev (Notices Amer. Math. Soc. **20** (1973), A22) had already shown that the answer to Tarski's question was negative even in the context of set theory with regularity.

[2] Permutation models can also be used for set theories whose axioms are those of ZF except that the axiom of regularity is weakened or eliminated. Specker deals with such a theory but his development of permutation models carries over verbatim for ZFU. We use ZFU rather than such a theory because it seems more natural.