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ON QUOTIENTS OF MANIFOLDS: A GENERALIZATION OF THE CLOSED SUBGROUP THEOREM

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Let M be a (C^∞, Hausdorff, paracompact) manifold, and let R be an equivalence relation on M. Then R is called *regular* if the quotient M/R is a (not necessarily Hausdorff) manifold in such a way that the canonical projection $\pi_R: M \to M/R$ is a submersion. For results on regular relations cf. Palais [**1**], Serre [**2**]. The following characterization of regularity is well-known (cf. Serre [**2**, LG, Chapter 3, §12]): R is regular if and only if it is a submanifold (with the subspace topology) of $M \times M$ in such a way that the map $(m, m') \to m$ from R onto M is a submersion.

The purpose of this note is to announce a different characterization of regularity. Proofs will appear elsewhere (Sussmann [**3**]).

Our condition is motivated in a natural way by Systems Theory. As will be shown in [**4**], Theorem 2 is precisely what is needed to show that, under fairly general conditions, every finite-dimensional "controllable" non-linear system has a realization which is both "controllable" and observable.

Here we shall not pursue this line. Rather, we shall state our condition and show that it is a rather natural generalization of the closed subgroup theorem.

Let X be a vector field on an open subset of M. We say that X is a *symmetry vector field of* R if, whenever $(m, m') \in R$, it follows that $(X_t(m), X_t(m')) \in R$ for every real t for which $X_t(m)$ and $X_t(m')$ are both defined (here $t \to X_t(m)$ is the integral curve of X which passes through m when $t=0$). Let $S^\infty(R, M)$ denote the set of all C^∞ vector fields X defined on open subsets of M that are symmetry vector fields for R. It is not difficult to show that $S^\infty(R, M)$ is a presheaf of Lie algebras of vector fields. If L is a set of vector fields defined on open subsets of M, we say that L is *transitive* if, for every $m \in M$, the vectors $X(m)$, $X \in L$, span the tangent space of M at m. If A is a subset of $M \times M$, we call L *A-transitive* if, for every $(m, m') \in A$, the tangent space of M at m is spanned by the

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