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CLOSED OPERATORS AND EXISTENCE THEOREMS IN MULTIDIMENSIONAL PROBLEMS OF THE CALCULUS OF VARIATIONS[1]

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1. **Weakly and strongly closed operators.** Let X be a topological space with topology τ (say, a Banach space with weak topology), let S be a subset of X, and Y a Banach space. An operator $\mathscr{A}:S\to Y$ (not necessarily linear) is said to be weakly [strongly] closed with respect to (X, τ) (briefly, $\mathscr{A}$ has property K_w $[K_\sigma]$) provided $x_k \in S$, $k=1, 2, \cdots$, $x \in X$, $y \in Y$, $x_k\to x$ in (X, τ), $\mathscr{A}x_k\to y$ weakly [strongly] in Y implies $x \in S$, $y=\mathscr{A}x$. Analogously, $\mathscr{A}:S\to Y$ is said to have the weak [strong] convergence property with respect to (X, τ) (briefly, $\mathscr{A}$ has property V_w $[V_\sigma]$) provided $x_k \in S$, $k=1, 2, \cdots, x \in S$, $x_k\to x$ in (X, τ) implies $\mathscr{A}x_k\to\mathscr{A}x$ weakly [strongly] in Y as $k\to\infty$.

Convergence properties V_w and V_σ have been already used in **[1]**, **[8]** toward existence theorems in multidimensional problems of optimization. The closure properties K_w and K_σ are used here in this context for the first time. These closure properties (closed graph properties) are well known. (See, e.g., N. Dunford and J. T. Schwartz **[11]** for linear operators; for nonlinear monotone operators see, e.g., G. J. Minty **[14]**.)

2. **Multidimensional problems of optimization with state equations in the strong form.** We are interested here in problems of optimization (Lagrange problems, or problems of optimal control) in a fixed bounded domain $G\subseteq E^\nu$. The unknown is an element of a Banach space X with norm $\|x\|$. State equations—in either strong or weak forms—and unilateral constraints are expressed in terms of general not necessarily linear operators on some subsets S of X, mapping S into vector-valued L-integrable functions on G and ∂G respectively, and of arbitrary measurable vector-valued control functions u on G and v on ∂G.

Precisely, we denote by Γ a closed subset of ∂G, by μ a suitable measure function on Γ, by T the set of all measurable vector functions $u(t)=(u^1, \cdots, u^m)$, $t \in G$ (distributed controls), and by $\mathring{T}$ the set of all μ-measurable vector functions $v(t)=(v^1, \cdots, v^{m'})$, $t \in \Gamma$ (boundary controls). We denote by $\mathscr{L}$, $\mathscr{J}$, $\mathscr{M}$, $\mathscr{K}$ given operators $\mathscr{L}:S\to(L_p(G))^r$,

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