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PSEUDO-INVERSES OF OPERATORS

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1. Let X and Y be complex Banach spaces, A a bounded linear operator from X to Y. If the null space $N(A)$ and the closed range $R(A)^-$ possess closed complementary subspaces U in X and V in Y respectively, the *pseudo-inverse* $A^\dagger$ *of* A *relative to* (U, V) is defined as the linear extension of $(A|U)^{-1}$ to $D(A^\dagger)=R(A)+V$ with the null space $N(A^\dagger)=V$. (This is a generalization to Banach space of the standard pseudo-inverse of a Hilbert space operator (cf. [**8**]). If $R(A)$ is closed, the definition agrees with the ones given in [**1**] and [7]. In this case $A^\dagger$ is defined and bounded on all of Y.) If $U=R(B)^-$ and $V=N(B)$ for some bounded linear operator $B: Y\to X$, $A^\dagger$ will be called the pseudo-inverse of A relative to B, written $A^{\dagger B}$. Proposition 6 of [**6**] leads to the following result.

THEOREM 1. *Suppose* $A:X\to Y$ *and* $B: Y\to X$ *are bounded linear operators such that* (a) $Y=R(A)^-\oplus N(B)$, (b) *the operator* $T=I-BA$ *is strongly power convergent* ($\{T^n\}$ *converges strongly*). *Then* $A^{\dagger B}$ *exists and is represented by*

$$A^{\dagger B}y = \sum_{n=0}^{\infty} (I - BA)^n By, \tag{1}$$

where the series converges in norm iff $y \in R(A)+N(B)$.

When T in Theorem 1 is uniformly power convergent ($\{T^n\}$ converges uniformly), then $R(A)$ is closed, (1) converges uniformly, and $A^{\dagger B}$ is defined and bounded on all of Y. In the case that A is an operator between Hilbert spaces, and $B=\alpha A^*$ with $0<\alpha<2\|A\|^{-2}$, Theorem 1 gives the well-known representation of the standard Hilbert space pseudo-inverse [**2**], [7], [**8**].

2. Let $A:X\to Y$ be a bounded linear operator between Banach spaces. A bounded linear operator $B: Y\to X$ is called a *pseudo-adjoint* of A if

$$X = N(A) \oplus R(B)^-, \qquad Y = R(A)^- \oplus N(B), \tag{2}$$

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