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HOMOTOPY IN HOMEOMORPHISM SPACES, *TOP* AND *PL*

BY MARY-ELIZABETH HAMSTROM

CHAPTER 0. BACKGROUND

0.1. **Introduction.** In 1926, Kneser published a paper **[39]** that received little notice until the 1950's. In this paper he proved, using conformal mapping theory, that the space $O(E^2)$ of rotations of the plane is a strong deformation retract of the space $Top(E^2)$ of all orientation preserving homeomorphisms of the plane onto itself, the topology for this space being the compact-open topology. Thus, the injection $i: O(E^2) \rightarrow Top(E^2)$ induces isomorphisms of the homotopy groups so that, as a consequence, the group of an E^2 bundle reduces to the orthogonal group.

In the 1950's, I tried to find conditions under which an open map $f: X \rightarrow I$, X compact metric, $I = [0, 1]$, each $f^{-1}(t)$ a 2-disc D^2, would be like the projection map of $D^2 \times I$ onto I. It turned out (Dyer-Hamstrom **[19]**) that the fact that the space of homeomorphisms of D^2 onto itself is locally contractible enabled us to apply a selection theorem of Michael to prove that f is like a projection map if it has certain regularity properties similar to equicontinuity (see § 0.3) and that if I is replaced by a finite dimensional separable metric space, f is like the projection map of a disc bundle. Thus, in addition to their intrinsic interest, solutions to problems concerning the homotopy groups of the space of all homeomorphisms on a manifold have important consequences in the study of fibre bundles and open mappings.

It is my purpose here to survey the state of the knowledge of homotopy properties of homeomorphism spaces and, since they are clearly related, the analogous properties of certain embedding spaces. (The reader is warned that space restrictions prevent my considering codimensions other than 1.) The remainder of this chapter is devoted to definitions and Michael's theorem. In Chapter I, I consider the topological category, in Chapter II, the PL category, and in Chapter III, PL approximations and the relationship between these categories.

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