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ON HILBERT TRANSFORMS ALONG CURVES

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Let $\gamma(t)$, $-\infty<t<\infty$, be a smooth curve in R^n. For f in $C_0^\infty(R^n)$ set

$$(1)\qquad Tf(x) = \lim_{\varepsilon\to\infty, N\to\infty} \int_{\varepsilon\leqq|t|\leqq N} \frac{f(x-\gamma(t))}{t}\,dt.$$

Tf is the Hilbert transform of f along the curve $\gamma(t)$. E. M. Stein [2] raised the following general question: For what values of p and what curves $\gamma(t)$ is Tf a bounded operator in L^p? If $\gamma(t)$ is a straight line it is well known that T is bounded for $1<p<\infty$. Stein and Wainger [3] proved that the operator is bounded for $p=2$ if

$$\gamma(t) = (|t|^{\alpha_1}\operatorname{sgn} t, \cdots, |t|^{\alpha_n}\operatorname{sgn} t), \qquad \alpha_i > 0.$$

Here we show that Tf is a bounded operator in L^p for some p other than 2 and some nontrivial, nonlinear γ's. We prove

THEOREM 1. *Let* $\gamma(t)=(|t|^{\alpha_1}\operatorname{sgn} t, |t|^{\alpha_2}\operatorname{sgn} t)\alpha_1>0$, $\alpha_2>0$. *Then* Tf *is bounded in* L^p *for* $\frac{4}{3}<p<4$.

SKETCH OF THE PROOF. The transformation (1) may be expressed as a multiplier transformation. In our case,

$$(2)\qquad (Tf)^\wedge(x, y) = m(x, y)\hat{f}(x, y)$$

where

$$(3)\quad m(x, y) = \lim_{\varepsilon\to\infty, N\to\infty} \int_{\varepsilon\leqq|t|\leqq N} \exp\{i\,|t|^{\alpha_1}\operatorname{sgn} tx + i\,|t|^{\alpha_2}\operatorname{sgn} ty\}\,\frac{dt}{t}$$

($\wedge$ denotes Fourier transform).

By a change of variables we may assume $\alpha_1=1$ and $\alpha_2\geqq 1$. Furthermore we may assume $\alpha_2>1$, for otherwise we have the case that $\gamma(t)$ is a straight

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