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A BOREL INVARIANTIZATION

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Let G be a nonmeager topological group having a countable basis, and suppose G acts on a topological space X. We show in §1 that for each Borel set $B \subseteq X$ there is an invariant Borel set B^* lying between the usual invariantizations

$$B^- = \{x : \forall g(gx \in B)\} \quad \text{and} \quad B^+ = \{x : \exists g(gx \in B)\}.$$

Moreover, the map $B \mapsto B^*$ has other regularities. Using §1, we establish in §2 invariant versions for some classical theorems. For example, we show: Every invariant analytic set is the union of $\aleph_1$ invariant Borel sets. The transform * is applied to infinitary logic in §3. We give a new proof of the theorem: "invariant Borel $= L_{\omega_1\omega}$", and obtain several new theorems of the same form.

A detailed presentation will appear in Fundamenta Mathematica.

1. **A transform.** We assume the action $(g, x) \mapsto gx \in X$ is continuous in each variable separately (and, as usual, that $ex = x$ and $g(hx) = (gh)x$). Suppose $\mathscr{H}$ is a family of nonempty open sets such that any nonempty open set includes a member of $\mathscr{H}$. (The existence of a countable $\mathscr{H}$ is a sufficient countability assumption.) Below "U", "V" always denote members of $\mathscr{H}$ and similarly $x \in X, g \in G, B \subseteq X, \alpha < \omega_1$, and $F : \omega \to \omega$. Ⓐ is the smallest family containing all open sets and closed under complement, countable union, and the operation (A). (For (A) and other notions below, see [3].)

DEFINITION 1.1. (a) $B^x = \{g : g^{-1}x \in B\}$.

(b) $B^* = \{x : B^x$ *is comeager in* $G\}$; *and in general*, $B^*_U = \{x : B^x \cap U$ *is comeager in* $U\}$.

THEOREM 1.2. (a) B^*_U *is closed if* B *is closed.*

(b) $(\bigcap_n B_n)^*_U = \bigcap_n (B_n)^*_U$.

(c) $(\sim B)^*_U = \sim \bigcup \{B^*_V : V \subseteq U\}$.

(d) $x \in (\bigcup_F \bigcap_n B_{F\restriction n})^*_U$ *if and only if*

$$(\forall U_0 \subseteq U)(\exists V_0 \subseteq U_0)(\exists k_0)$$
$$(\forall U_1 \subseteq V_0)(\exists V_1 \subseteq U_1)(\exists k_1) \cdots \forall n[x \in (B_{k_0\cdots k_n})^*_{V_n}].$$

The left side of (d) is the operation (A), while the right side denotes an

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