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INDEX THEORY FOR SINGULAR QUADRATIC FUNCTIONALS IN THE CALCULUS OF VARIATIONS[1]

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1. **Introduction.** Let P, Q, and R be real-valued $n \times n$ matrix functions defined on the interval $[a, b)$. Assume that P, Q, and R are continuous on $[a, b)$ and that $P(t)$ and $R(t)$ are symmetric matrices for each t in $[a, b)$. We do not assume that Q is symmetric. Also assume that R has the property that its value for any t in $[a, b)$ is positive definite, that is, $v^*R(t)v > 0$ for all n-vectors $v \neq 0$ and for each t in $[a, b)$. Let

$$J(x, y)\,\Big|_{e_1}^{e_2} = \int_{e_1}^{e_2} [\dot{x}^*(t)R(t)\dot{y}(t) + x^*(t)Q(t)\dot{y}(t) + \dot{x}^*(t)Q^*(t)y(t) + x^*(t)R(t)y(t)]\,dt \qquad (a \leqq e_1 \leqq e_2 < b), \tag{1.1}$$

for x and y in the class A of vector-valued functions described below. Also let

$$J_e(x, y) = J(x, y)\,\Big|_a^e, \qquad J_e(x) = J_e(x, x), \tag{1.2}$$

$$J(x, y) = \liminf_{e \to b-} J_e(x, y), \qquad J(x) = \liminf_{e \to b-} J_e(x) \tag{1.3}$$

for x and y in A. The class A is the set of vector-valued functions $x^*(t) = (x_1(t), \ldots, x_n(t))$, $a \leqq t \leqq b$, satisfying

(i) $x(t)$ is continuous on the interval $[a, b]$ and $x(a) = x(b) = 0$,

(ii) $x(t)$ is absolutely continuous and $\dot{x}^*(t)\dot{x}(t)$ is Lebesgue integrable on each closed subinterval of $[a, b)$. A is a vector space of functions.

J is said to be *singular* at a point t in $[a, b]$ if the determinant of $R(t)$ is zero or not defined. The point $t = b$ is a singular point in this paper.

2. **Preliminaries.** What is presented here is part of a quadratic form theory developed and used extensively by Hestenes [3], [4]. Let $Q(x)$

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