BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 5, September 1973

SOME TOPOLOGICAL APPROACHES TO THE FOUR COLOR PROBLEM

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Communicated by Gian-Carlo Rota, March 15, 1973

The aim of this paper is to give a few observations with no proofs about the four color problem that are topological in nature. The approach is to define a 'local' 4 coloring on any 2-manifold and to measure the obstructions to a local coloring's being a desired type of coloring. All manifolds will be assumed to be oriented.

1. **Local colorings.** We begin with the definitions of four types of coloring that are possible on any surface.

(1) *A four coloring* of a triangulation K is a map $f: K \to \partial\Delta^3$ which is simplicial, and maps triangles onto triangles. Here $\partial\Delta^3$ is the tetrahedron, which has exactly 4 vertices.

(2) An *edge coloring* of K is a division of the edges of K into 3 sets, so that every triangle of K has an edge in each set. If we think of the sets as colors, every triangle has its edges 3 different colors.

(3) A *heawood coloring* is an assignment of $+1$ or -1 to every triangle of K so that the sum of the values on the triangles containing a point p is zero mod 3 for all vertices p of K.

(4) A *local coloring* is a collection of maps $f_p: st(p) \to \partial\Delta^3$ and automorphisms σ_{pq} of $\partial\Delta^3$ so that the following commutes.

$$\begin{array}{ccc} st(p) & \cap & st(q) \\ {\scriptstyle f_p}\swarrow & & \searrow{\scriptstyle f_q} \\ \partial\Delta^3 & \xrightarrow[\sigma_{pq}]{} & \partial\Delta^3 \end{array}$$

In this definition, $st(p)$ is the complex consisting of all triangles (and faces of them) containing p. σ_{pq} can be given as a permutation of the 4 vertices of $\partial\Delta^3$. We can restate the definition by saying we have a 4 coloring for each $st(p)$ and, up to permutation of colors, the colorings of $st(p)$ and $st(q)$ agree on their overlap.

Next, we show how each type of 'coloring' induces the following one. Let $f: K \to \partial\Delta^3$ be a four coloring. $\partial\Delta^3$ has an edge coloring, so we get an edge coloring for K by pulling back the one on $\partial\Delta^3$. That is, the color of

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AMS (*MOS*) *subject classifications* (1970). Primary 55A15; Secondary 05C15, 05C10.