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FUNDAMENTAL GROUPS, NILMANIFOLDS AND ITERATED INTEGRALS[1]

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Let X be a connected C^∞ manifold. Denote by $P(X)$ the total space of piecewise smooth paths in X. Choose a base point x_0. Denote by $P(X; x_0)$ (resp. ΩX) the space of piecewise smooth paths (resp. loops) from the base point x_0.

Let k be the field of real (or complex) numbers. All differential forms are k-valued. Let $w_1, w_2, \ldots$ denote 1-forms on X. For a piecewise smooth path $\alpha: I \to X$, let $f_i(t) = w_i(\alpha(t), \dot\alpha(t))$ be the value of the 1-form w_i at the tangent vector $\dot\alpha(t)$ of X. Define the r-time iterated integral $\int w_1 \cdots w_r$ to be the k-valued function on $P(X)$ whose value at α is given by

$$\left\langle \int w_1 \cdots w_r, \alpha \right\rangle = \int_0^1 \int_0^{t_r} \cdots \int_0^{t_2} f_1(t_1)\, dt_1 \cdots f_{r-1}(t_{r-1})\, dt_{r-1} f_r(t_r)\, dt_r$$

when $r > 0$ and $= 1$ when $r = 0$. At times, we shall also take $\int w_1 \cdots w_r$ as its restriction on ΩX or $P(X; x_0)$.

Let F be the function algebra on $P(X)$ consisting of those functions whose value at each path α remains invariant under any piecewise smooth homotopy of α relative to $\dot I$. In this note, we shall consider the subspace of F whose elements are linear combinations of iterated integrals. A characterization of this subspace in terms of the fundamental group $\pi_1(X)$ will be given.

We begin with a differential graded subalgebra A of the exterior algebra $\Lambda(X)$. The following assumptions are made:

I. $dA^0 = A^1 \cap d\Lambda^0(X)$.

II. $\dim H^1(A) < \infty$.

III. The canonical homomorphism $H^q(A) \to H^q(X; k)$ is an isomorphism when $q = 1$ and is a monomorphism when $q = 2$.

A primary example is the case of $A = \Lambda(X)$.

For $s \geqq 0$, denote by $F_A(s)$ the subspace of F whose elements are linear combinations of iterated integrals of the type

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