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EXACT COLIMITS

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It is well known and easy that if $\boldsymbol{C}$ is a small category with filtered components, then the functor $\text{colim}_{\boldsymbol{C}}:\text{Ab}^{\boldsymbol{C}} \to \text{Ab}$ is exact. The converse was conjectured and proved in a special case by Oberst [**4**]. A necessary and sufficient condition for exactness of $\text{colim}_{\boldsymbol{C}}$ was given by Isbell in [**2**], who used the condition to show that Oberst's conjecture is true when $\boldsymbol{C}$ is a monoid. We show that the conjecture is false in general. Proofs will only be sketched here, full details to appear elsewhere.

1. **Affinization.** If A and B are objects of $\boldsymbol{C}$, then *A maps to B* if $\boldsymbol{C}(A, B)$ is nonempty. If α_i is a family of $\boldsymbol{C}(A, B)$, then *β filters* the family if $\beta\alpha_i$ is independent of i. A category $\boldsymbol{C}$ is *filtered* if every pair (and hence every finite family) of objects map to a common object, and every pair (and hence every finite family) of morphisms with common domain and codomain are filtered.

The *additivization* of $\boldsymbol{C}$ is the category $\boldsymbol{ZC}$ with the same objects, where $\boldsymbol{ZC}(A, B)$ is the free abelian group on $\boldsymbol{C}(A, B)$. The *affinization* of $\boldsymbol{C}$ is the subcategory of $\boldsymbol{ZC}$ of morphisms whose integer coefficients sum to one. Note that $\boldsymbol{C} \subset \text{aff}\,\boldsymbol{C}$, with equality if and only if $\boldsymbol{C}$ is a preordered set.

If $M \in \text{Ab}^{\boldsymbol{C}}$, then $\text{colim}_{\boldsymbol{C}} M = \bigoplus_{A \in |\boldsymbol{C}|} M(A)/X$ where X is the subgroup of the numerator generated by elements of the form $x - \alpha x$ with, say, $x \in M(A)$, $\alpha \in \boldsymbol{C}(A, B)$, and hence $\alpha x \in M(B)$. Note that if $\sum n_i\alpha_i$ is a morphism of $\text{aff}\,\boldsymbol{C}$, then

$$x - (\textstyle\sum n_i\alpha_i)x = \sum n_i(x - \alpha_i x),$$

and it follows that if M is considered as an object of $\text{Ab}^{\text{aff}\,\boldsymbol{C}}$ in the obvious way, then $\text{colim}_{\boldsymbol{C}} M = \text{colim}_{\text{aff}\,\boldsymbol{C}} M$. This yields easily the "if' part of the following theorem, which is close to being a restatement of [**2**, Theorem 1].

THEOREM 1. $\text{Colim}_{\boldsymbol{C}}$ *is exact if and only if the components of* $\text{aff}\,\boldsymbol{C}$ *are filtered.*

The converse is an application of the "several object" version of ring theory [**3**]. We express the colimit $\cdots$ as $\text{colim}_{\boldsymbol{C}} M = \Delta\boldsymbol{Z} \otimes_{\boldsymbol{ZC}} M$ where $\Delta\boldsymbol{Z}$ is the constant functor at $\boldsymbol{Z}$ over $\boldsymbol{C}^{\text{op}}$. Then exactness of $\text{colim}_{\boldsymbol{C}}$ is

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