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PERIODIC SOLUTIONS OF AUTONOMOUS FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The purpose of this note is to indicate some applications of a new fixed point theorem to the question of periodic solutions of nonlinear autonomous functional differential equations. The techniques developed give the standard periodicity examples in the literature and some new results, notably for the neutral case, which do not seem accessible by previous methods.

1. If X is a Banach space and A is a bounded subset of X, define $\gamma(A)$, the measure of noncompactness of A, to be $\inf\{d > 0 : A$ has a finite covering by sets of diameter less than $d\}$. This is a notion due to C. Kuratowski [**13**]. G. Darbo observed [**4**] that if $\overline{\text{co}}(A)$ denotes the convex closure of a set A and if $A + B = \{a + b : a \in A, b \in B\}$ for sets A and B, then (1) $\gamma(\overline{\text{co}}(A)) = \gamma(A)$ and (2) $\gamma(A + B) \leqq \gamma(A) + \gamma(B)$. It is trivially true that (3) $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$.

For applications it is sometimes convenient to generalize the above idea slightly. If μ is a function which assigns to each bounded subset A of X a real number $\mu(A)$, we say that μ is a generalized measure of noncompactness if μ satisfies properties (1), (2) and (3) above and if there exist positive constants m and M such that $m\mu(A) \leqq \gamma(A) \leqq M\mu(A)$ for every set $A \subset X$. If J is a closed bounded interval of $\boldsymbol{R}$ and $C^1(J, \boldsymbol{R}^n)$ denotes the Banach space of continuously differentiable maps from J to $\boldsymbol{R}^n$ with any of the standard norms, then if

$$\mu(A) = \lim_{\delta \to 0; \delta > 0} (\sup\{|x'(t) - x'(s)| : x \in A, t, s \in J, |t - s| < \delta\}),$$

μ is an example of a generalized measure of noncompactness on $C^1(J, \boldsymbol{R}^n)$.

If U is a subset of a Banach space X, $f : U \to X$ is a continuous map, and μ is a generalized measure of noncompactness, then we shall say that f is a k-set-contraction with respect to μ if for every bounded set $A \subset U$, $f(A)$ is bounded and $\mu(f(A)) \leqq k\mu(A)$. If G is a closed, convex subset of X, U is a bounded open subset of G, and $f : \overline{U} \to G$ is a k-set-contraction with respect to μ, $k < 1$, then if $f(x) \neq x$ for $x \in \overline{U} - U$, there is an integer defined, called the fixed point index of f on U and written $i_G(f, U)$. Details are given in [**15**], where the fixed point index is actually defined for a larger class of maps which are defined on open subsets of certain metric

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