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EXISTENCE OF THE STABLE HOMOTOPY FAMILY $\{\gamma_t\}$

BY RAPHAEL ZAHLER

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Toda [7] has asked whether Smith's $V(n)$-construction for $n = 3$ yields a nontrivial element γ_1 in the p-component ${}_p\pi_*^S$ of the stable homotopy of spheres (p a prime, $p \geqq 5$).[1] This question has become a major stumbling block, since γ_1 has stubbornly refused to be detected by most conventional invariants [9]. We can now show that γ_1 is essential; moreover (for $p \geqq 7$) it is only the first of a new family $\{\gamma_t\}$ of stable homotopy elements, which are nontrivial for $t \leqq p - 1$ at least. The family $\{\gamma_t\}$ parallels the known infinite families $\{\alpha_t\}$ and $\{\beta_t\}$ ([1], [4], [8], [10], [12]).

We define γ_t to be the composite

$$S^{2t(p^3-1)} \hookrightarrow S^{2t(p^3-1)}V(2) \xrightarrow{\chi_t} V(2) \to S^{2p^2+2p-1},$$

in the stable category, where the $V(n)$ are the spectra introduced by Smith ([6], [8]), $\chi : S^{2(p^3-1)}V(2) \to V(2)$ is a map whose cone is $V(3)$, and χ_t is the usual iterate of suspensions of χ. The map χ is known to exist only for $p \geqq 7$, but a similar construction defines γ_1 for $p = 5$ as well [7].

THEOREM A. *The element* $\gamma_1 \in {}_p\pi^S_{(p^2-1)q-3}$ ($p \geqq 5$, $q = 2(p-1)$) *is essential.*

Since it is known that ${}_p\pi^S_{(p^2-1)q-3} \cong Z_p$, generated by $\alpha_1\beta_{p-1}$ [4], γ_1 must be a nonzero multiple of $\alpha_1\beta_{p-1}$. Thus Theorem A does not exhibit a new stable homotopy element; rather, it shows that the first element produced by the $V(n)$ construction is nontrivial.

COROLLARY.

$$\left.\begin{aligned} \alpha_1\beta_{p-1}\beta_s &= 0, && s \geqq 3, \\ \alpha_1\beta_1\beta_k &= 0, && k \not\equiv -2 \mod p, k \geqq p, \\ \alpha_1\beta_2\beta_{k-1} &= 0, && k \not\equiv -2 \mod p, k \geqq p+1, \end{aligned}\right\} p \geqq 5.$$

This follows from Theorem A and Proposition 5.9 of [7].

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[1] The element $\gamma_1 \in {}_p\pi^S_{(p^2-1)q-3}$ should not be confused with the ephemeral element $\gamma \in {}_p\pi^S_{p^2q-2}$ whose nonexistence was proved by Toda [5].