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# GEODESICS OF SINGULAR RIEMANNIAN METRICS

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1. **Introduction.** In this note, I shall briefly present a geometric notion, which I believe is new, and which may have interesting applications to physics (e.g. in the nonrelativistic limit of cosmological models) and to mathematics (e.g. the study of the boundary properties of the Bergmann metric on domains in complex Euclidean space). More details will appear in a forthcoming book [**6**].

What I shall do is to reformulate the notion of "geodesic" for a nonsingular Riemannian metric so that it makes sense in the singular case. This will require that the reader be familiar with the notion of the "Hamilton-Jacobi equation" associated with a variational problem. My book [**1**] can be used as a reference for this material and for the notations used in this note.

2. **Geodesics of nonsingular Riemannian metrics.** Let $M$ be a $C^\infty$ manifold. $V(M)$ denotes its $C^\infty$ vector fields, $F(M)$ its $C^\infty$, real-valued functions, and $F^1(M)$ its differential forms of degree one.

A *Riemannian metric* is usually defined as an $F(M)$-bilinear symmetric map $\beta: V(M) \times V(M) \to F(M)$ which is nondegenerate. Such a $\beta$ defines an $F(M)$-linear isomorphism $\alpha: F^1(M) \to V(M)$ with the following property:

$$\alpha^{-1}(X)(Y) = \beta(X, Y) \quad \text{for} \quad X, Y \in V(M). \tag{2.1}$$

Let $\beta^d$ be the form: $F^1(M) \times F^1(M) \to F(M)$ defined as follows:

$$\beta^d(\theta_1, \theta_2) = \beta(\alpha(\theta_1), \alpha(\theta_2)) \quad \text{for} \quad \theta_1, \theta_2 \in F^1(M). \tag{2.2}$$

DEFINITION. A function $f \in F(M)$ is a *Hamilton-Jacobi function* of the metric if there is a function $h(\ )$ of one variable such that:

$$\beta^d(df, df) = h(f). \tag{2.3}$$

Let $\operatorname{grad}: F(M) \to V(M)$ be the first order linear differential operator defined as follows:

$$\operatorname{grad} f = \alpha(df). \tag{2.4}$$

THEOREM 2.1. *A curve in $M$ is a geodesic of the nonsingular metric $\beta$ if and only if there exists, locally, a Hamilton-Jacobi function $f$ such that the curve is an integral curve of* $\operatorname{grad} f$.

The proof of Theorem 2.1 is given in [**1**], and is a consequence of classical

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